Analysis of a Variable Time-step Discretization of the Three-Dimensional Frémond Model for Shape Memory Alloys*

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Abstract

This paper deals with a semi-implicit time discretization with variable step of a three-dimensional Frémond model for shape memory alloys. Global existence and uniqueness of a solution is discussed. Moreover, an *a priori* estimate for the discretization error is recovered. The latter depends solely on data, imposes no constraints between consecutive time steps, and shows an optimal order of convergence when referred to a simplified model.

Key words: shape memory alloys, time discretization, existence and uniqueness, error estimate.

AMS (MOS) Subject Classification: 80A22, 35K55, 65M15

1 Introduction

This paper is concerned with the following system of partial differential equations in terms of the unknown functions ϑ, χ_1, χ_2, and $\boldsymbol{u}$

$$\partial_t\big(c_0\vartheta - L\chi_1\big) \;+\; \partial_t\big((\alpha(\vartheta)-\vartheta\alpha'(\vartheta))\chi_2 \operatorname{div}\boldsymbol{u}\big) \;-\; h\Delta\vartheta \;=\; F, \tag{1.1}$$

$$\operatorname{div}\left(-\nu\Delta(\operatorname{div}\boldsymbol{u})J + \lambda \operatorname{div}\boldsymbol{u}J + 2\mu\varepsilon(\boldsymbol{u}) + \alpha(\vartheta)\chi_2 J\right) + \boldsymbol{G} \;=\; \boldsymbol{0}, \tag{1.2}$$

$$k\partial_t\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \;+\; \begin{pmatrix}\ell(\vartheta-\vartheta^*)\\ \alpha(\vartheta)\operatorname{div}\boldsymbol{u}\end{pmatrix} \;+\; \partial I_K(\chi_1,\chi_2) \;\ni\; \begin{pmatrix}0\\0\end{pmatrix}, \tag{1.3}$$

a.e. in $Q := \Omega\times(0,T)$ where Ω is a bounded open subset of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and $T>0$ stands for some final time. In addition, c_0, L, h, λ, μ, k, ℓ, and ϑ^* are positive parameters, J is the identity matrix in $\mathbb{R}^3$, and ν is a nonnegative constant. Here, ε denotes the tensor

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{for } i,j=1,2,3, \tag{1.4}$$

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while $\partial I_{\mathcal{K}}$ stands for the subdifferential of the indicator function of a nonempty, bounded, convex and closed subset $\mathcal{K}$ of $\mathbb{R}^2$, and $\alpha : \mathbb{R} \to \mathbb{R}$, $F : Q \to \mathbb{R}$, $\boldsymbol{G} : Q \to \mathbb{R}^3$ are given functions with some properties to be specified later.

The nonlinear system (1.1)-(1.3) is concerned with the behavior of *shape memory alloys* subject to thermo-mechanical treatments. These materials are metallic alloys which could be permanently deformed (avoiding fractures) and consequently be forced to recover the original shape just by thermal means.

In the microscopic scale, this phenomenon is interpreted as the effect of a structural phase transition between different configurations of the metallic lattices, namely the *austenite* and its shared counterparts termed *martensites* (see, e.g., [**?**]). Various models have been proposed to describe this behavior from the macroscopic point of view (see [**?**]). If we assume the phases to coexist at each point of the shape memory sample and suppose that just two martensitic variants are present besides one austenite (in the three-dimensional space, up to 24 martensitic variants have been detected), indeed we may deal with the approach proposed by Frémond [**?**, **?**, **?**, **?**]. In this context, ϑ has to be regarded as the absolute temperature of the shape memory body while $\boldsymbol{u}$ accounts for its actual displacement and ε stands for the (linearized) strain tensor. Besides, $\alpha(\vartheta)$ represents the thermal expansion of the system, and thus it vanishes at high temperatures (cf., e.g., [**?**, **?**]). In our analysis α is also required to fulfill some compatibility conditions complying with the physical setting (see [**?**, **?**]). Regarding the phases, let β_1, β_2, and β_3 be the volumetric proportions of the two martensitic variants and of the austenite, respectively. These quantities obviously fulfill the conditions

$$\beta_1 + \beta_2 + \beta_3 = 1, \quad 0 \le \beta_i \le 1 \quad \text{for } i = 1, 2, 3. \tag{1.5}$$

Defining the variables χ_1 and χ_2 as

$$\chi_1 := \beta_1 + \beta_2, \qquad \chi_2 := \beta_1 - \beta_2,$$

relation (1.5) implies that

$$[\chi_1, \chi_2] \in K := \Big\{ [\gamma_1, \gamma_2] \in \mathbb{R}^2 \text{ such that } |\gamma_2| \le \gamma_1 \le 1 \Big\}. \tag{1.6}$$

From the constitutive laws coupled with the second principle of thermodynamics and the universal conservation laws for momentum and energy, one deduces the system (1.1)-(1.3). Note that equation (1.2) is considered in a *quasi-stationary* form, that is, the inertial term $\boldsymbol{u}_{tt}$ is omitted. Indeed, let us stress that the latter *small deformations* approximation of the momentum balance equation is a rather standard approach [**?**, **?**, **?**, **?**, **?**, **?**, **?**]. Moreover, note that the existence of a solution to the three-dimensional problem with full momentum and nonlinearities is still an open and extremely challenging question (the reader is referred to [**?**] where the full momentum equation is considered along with a linearized energy balance equation).

On the other hand, we stress that the energy balance equation of the full Frémond model [**?**] turns out to be

$$\partial_t\big(c_0\vartheta - L\chi_1\big) + \partial_t\big((\alpha(\vartheta) - \vartheta\alpha'(\vartheta))\chi_2 \operatorname{div} \boldsymbol{u}\big) - h\Delta\vartheta = F + \alpha(\vartheta)\chi_2\, \partial_t\big(\operatorname{div} \boldsymbol{u}\big), \tag{1.7}$$

while, in our framework, the nonlinearity in the right hand side of the previous equation is neglected. This simplification of the model has a technical motivation and seems mandatory in order to perform some error analysis. Indeed, from the analytical point of view, the choice of considering (1.1) instead of (1.7) is strictly connected with the crucial possibility of establishing a global in time error estimate, i.e. up to any reference time T. As regards the physical viewpoint, it is well known that the quantity $\|\alpha\|_{L^\infty(\mathbb{R})}$ turns out to be very small with respect to the other data whenever a real alloy is taken into account [**?**]. In this connection, a reasonable simplification of the model would be that of linearizing completely the energy balance equation (1.7). The latter was exactly the original approach to the model proposed and investigated in the paper [**?**] and we may find in the literature some contributions dealing just with some of the nonlinearities of (1.7) ([**?**, **?**]). On the other hand, we shall remark that the model (1.1)-(1.3) is still suitable of describing completely the effect of the phase transition on the energy balance equation and that our simplification consists in neglecting part of the mechanically induced heat sources.

Finally, let us refer the reader to [**?**] for the physical meaning of the constants c_0, L, h, ν, λ, μ, k, ℓ, and ϑ^*.

The system (1.1)-(1.3) has to be supplied with suitable initial and boundary conditions. We prescribe

$$\vartheta(\cdot,0)=\vartheta_0,\quad \chi_1(\cdot,0)=\chi_{1,0},\quad \chi_2(\cdot,0)=\chi_{2,0}. \tag{1.8}$$

Denoting by $\partial_{\boldsymbol{n}}$ the outward derivative to the boundary $\partial\Omega$ and letting $\{\Gamma_0,\Gamma_{\mathcal{N}}\}$ be a partition of $\partial\Omega$ into measurable subsets with positive surface measures, we choose

$$\begin{aligned} h\partial_{\boldsymbol{n}}\vartheta+\eta(\vartheta-f) &= 0 \quad \text{on } \partial\Omega\times(0,T), && (1.9)\\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma_0\times(0,T), && (1.10)\\ \big((-\nu\Delta(\operatorname{div}\boldsymbol{u})+\lambda\operatorname{div}\boldsymbol{u}+\alpha(\vartheta)\chi_2)J+2\mu\varepsilon(\boldsymbol{u})\big)\boldsymbol{n} &= \boldsymbol{g} \quad \text{on } \Gamma_{\mathcal{N}}\times(0,T), && (1.11)\\ \partial_{\boldsymbol{n}}(\operatorname{div}\boldsymbol{u}) &= 0 \quad \text{on } \partial\Omega\times(0,T). && (1.12) \end{aligned}$$

Here η denotes a positive parameter while $f:\partial\Omega\times(0,T)\to\mathbb{R}$, $\boldsymbol{g}:\Gamma_{\mathcal{N}}\times(0,T)\to\mathbb{R}^3$ account for the interaction with the medium surrounding the domain.

Existence of solutions to various problems concerning systems close to (1.1)-(1.3) is well known (see [**?**] for a review). Nevertheless, to our knowledge, an existence result for (1.1)-(1.3) was not yet investigated. In this concern, this paper provides the global existence and the uniqueness of a solution. Note that, the question whether or not the full problem (thus keeping (1.7) instead of (1.1)) has a unique solution has already been positively solved in the paper [**?**].

On account of the present literature on this model, we notice that the existence of solutions to systems related to the Frémond model rely often on a suitable *time-discretization – a priori estimates – passage to the limit* procedure. In this direction, the main novelty of the present contribution is that of proving an *optimal order a priori* estimate of the discretization error of (a variable step version of) such an approximation. This estimate depends solely on data. In particular, the latter estimate is independent of the regularity of the continuous solution. Moreover, no constraint between consecutive time steps are imposed throughout the analysis of the approximation.

As regards the error analysis of the nonlinear inclusion (1.3) we shall remark that our technique is not new. Indeed, our argument relies on a careful application of the abstract analysis devised and fully detailed in [**?**, **?**].

Let us point out that a parallel investigation of the discretization error for the one-dimensional Frémond model for shape memory alloys is carried out in [**?**]. In the latter paper we prove an optimal order error estimate for the one-dimensional version of the full model (1.2)-(1.3), (1.7), thus retaining all the nonlinearities in the energy balance equation. We shall stress that the error analysis of the one-dimensional case is entirely different from the present one and relies deeply in the 1-D structure of the problem. In particular, we make a crucial use of the possibility of rewriting an equivalent formulation of the problem which turns out to be completely independent of u_x.

This is the plan of the paper. In Section 2 we give a variational formulation of the continuous problem (1.1)-(1.3), (1.8)-(1.12). Section 3 contains the approximation and the statement of our main results. The existence of a solution to the system (1.1)-(1.3) is proved in Section 4, while Section 5 is devoted to deduce its uniqueness. Finally, Section 6 brings to the proof of the error estimate.

2 Continuous Problem

We start by fixing some notations. Let $(\cdot,\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in $L^2(\Omega)$, respectively, while $[\cdot,\cdot]$ stands for the general pair. We introduce the following Hilbert space

$$\boldsymbol{V} := \{\boldsymbol{v} \in (H^1(\Omega))^3, \text{ such that } \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0, \ \operatorname{div} \boldsymbol{v} \in H^1(\Omega)\},$$

endowed with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{V}} := \left(\nu \int_\Omega |\nabla(\operatorname{div} \boldsymbol{v})|^2 + \sum_{i=1}^3 \int_\Omega |\nabla v_i|^2\right)^{1/2}, \quad \boldsymbol{v} = (v_1, v_2, v_3) \in \boldsymbol{V}. \tag{2.1}$$

We also set, for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$,

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \left(\nu \nabla(\operatorname{div} \boldsymbol{u}) \cdot \nabla(\operatorname{div} \boldsymbol{v}) + \lambda \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} + 2\mu \sum_{i,j=1}^3 \varepsilon_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\right), \tag{2.2}$$

where ε stands for the strain tensor specified in (1.4). It is well known (see, e.g. [**?**, p. 110]) that there exists a positive constant $c_{\boldsymbol{V}}$ depending on λ, μ and Ω such that

$$a(\boldsymbol{v}, \boldsymbol{v}) \ \geq \ c_{\boldsymbol{V}} \|\boldsymbol{v}\|_{\boldsymbol{V}}^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{2.3}$$

Moreover, it is not difficult to verify that

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq \nu \|\nabla(\operatorname{div} \boldsymbol{v})\|_{(L^2(\Omega))^3}^2 + (\lambda + 2\mu/3)\|\operatorname{div} \boldsymbol{v}\|^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{2.4}$$

Since the special triangular form of $\mathcal{K}$ specified by relation (1.6) is not needed for our analysis, let $\mathcal{K}$ be an arbitrary nonempty, bounded, convex, and closed subset of $\mathbb{R}^2$, and define the (convex and closed) set

$$K := \big\{[\gamma_1, \gamma_2] \in (L^2(\Omega))^2, \text{ such that } [\gamma_1, \gamma_2] \in \mathcal{K} \text{ a.e. in } \Omega\big\}. \tag{2.5}$$

It is now straightforward to fix a positive constant $c_{\mathcal{K}}$ such that

$$\Big(|\gamma_1(x)|^2 + |\gamma_2(x)|^2\Big)^{1/2} \le c_{\mathcal{K}} \quad \forall\, [\gamma_1, \gamma_2] \in K, \quad \text{for a.e. } \; x \in \Omega. \tag{2.6}$$

We assume that the data fulfill

$$F \in L^2(Q), \tag{2.7}$$
$$f \in W^{1,1}(0, T; L^2(\partial\Omega)), \tag{2.8}$$
$$\boldsymbol{G} \in H^1(0, T; (L^2(\Omega))^3), \tag{2.9}$$
$$\boldsymbol{g} \in H^1(0, T; (L^2(\Gamma_{\mathcal{N}}))^3), \tag{2.10}$$
$$\vartheta_0 \in H^1(\Omega), \quad [\chi_{1,0}, \chi_{2,0}] \in K, \tag{2.11}$$

and ask α to be a smooth function, vanishing in the interval $(\vartheta_c, +\infty)$, where $\vartheta_c > 0$ stands for the so called Curie temperature. Moreover, we require that

$$\alpha \in C^2(\mathbb{R}) \;\text{ and the set }\; \{\xi \in \mathbb{R} : \alpha'(\xi) \neq 0\} \;\text{ is contained in }\; [0, \vartheta_c], \tag{2.12}$$

and

$$c_\alpha := \|\alpha''\|_{L^\infty(\mathbb{R})} \;\text{ is sufficiently small.} \tag{2.13}$$

The previous condition will be made precise in the sequel (see (2.25)-(2.26)) and is satisfied by physically realistic data.

We stress that, (2.12) ensures the validity of the inequalities

$$|\alpha'(\xi)|, |\xi\alpha''(\xi)| \le \vartheta_c c_\alpha, \quad |\alpha(\xi)|, |\xi\alpha'(\xi)| \le \vartheta_c^2 c_\alpha \qquad \forall \xi \in \mathbb{R} \tag{2.14}$$

where c_α is defined as above.

Remark 2.1. We note that some properties of α such as monotonicity (in the sense that α is a decreasing function) and positiveness, although physically motivated (see [**?**]), are not used in our analysis.

For the sake of convenience and owing to (2.3), (2.9)-(2.12) and to the Lax-Milgram lemma, we introduce the initial displacement $\boldsymbol{u}_0 \in \boldsymbol{V}$ defined as the unique solution of the variational equality corresponding to the initial values, namely

$$a(\boldsymbol{u}_0, \boldsymbol{v}) + \big(\alpha(\vartheta_0)\chi_{2,0}, \operatorname{div} \boldsymbol{v}\big) = \int_\Omega \boldsymbol{G}(\cdot, 0) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_{\mathcal{N}}} \boldsymbol{g}(\cdot, 0) \cdot \boldsymbol{v}\, d\Gamma \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{2.15}$$

Thus, a precise formulation of problem (1.1)-(1.3), (1.8)-(1.12) is the following

Problem (P). Find $\vartheta \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$, $\boldsymbol{u} \in H^1(0,T;\boldsymbol{V})$, $\chi_1, \chi_2 \in H^1(0,T;L^2(\Omega))$ such that

$$\operatorname{div}\boldsymbol{u} \in C^0(\overline{Q}), \tag{2.16}$$

and the following equations and conditions hold

$$[\chi_1(\cdot,t), \chi_2(\cdot,t)] \in K, \ \forall t \in [0,T], \tag{2.17}$$

$$\begin{aligned} &\Big(\partial_t(c_0\vartheta - L\chi_1), \varphi\Big) + \Big(\partial_t\big((\alpha(\vartheta) - \vartheta\alpha'(\vartheta))\chi_2 \operatorname{div}\boldsymbol{u}\big), \varphi\Big) \\ &\quad + h\int_\Omega \nabla\vartheta\cdot\nabla\varphi\,dx + \eta\int_{\partial\Omega}(\vartheta - f)\varphi\,d\Gamma = (F,\varphi) \\ &\qquad \forall\varphi \in H^1(\Omega), \quad \text{a.e. in } (0,T), \end{aligned} \tag{2.18}$$

$$\begin{aligned} &a(\boldsymbol{u},\boldsymbol{v}) + \Big(\alpha(\vartheta)\chi_2, \operatorname{div}\boldsymbol{v}\Big) = \int_\Omega \boldsymbol{G}\cdot\boldsymbol{v}\,dx + \int_{\Gamma_\mathcal{N}} \boldsymbol{g}\cdot\boldsymbol{v}\,d\Gamma \\ &\qquad \forall\,\boldsymbol{v} \in \boldsymbol{V}, \ \text{ a.e. in } (0,T), \end{aligned} \tag{2.19}$$

$$\begin{aligned} &k\sum_{j=1}^2 \Big(\partial_t\chi_j, \chi_j - \gamma_j\Big) + \Big(\ell(\vartheta - \vartheta^*), \chi_1 - \gamma_1\Big) \\ &\quad + \Big(\alpha(\vartheta)\operatorname{div}\boldsymbol{u}, \chi_2 - \gamma_2\Big) \le 0 \qquad \forall\,(\gamma_1,\gamma_2) \in K, \ \text{ a.e. in } (0,T), \end{aligned} \tag{2.20}$$

$$\vartheta(\cdot,0) = \vartheta_0 \quad \text{ a.e. in } \Omega. \tag{2.21}$$

$$[\chi_1,\chi_2](\cdot,0) = [\chi_{1,0},\chi_{2,0}] \quad \text{ a.e. in } \Omega. \tag{2.22}$$

An early result for Problem **(P)** is the following (see [**?**, Lemma 1])

Lemma 2.2. *For any* $\vartheta, \chi_2 \in C^0([0,T];L^2(\Omega))$ *satisfying*

$$|\chi_2(\cdot,t)| \le c_\mathcal{K} \quad \text{a.e. in } \Omega, \ \forall t \in [0,T], \tag{2.23}$$

there exists one and only one solution $\boldsymbol{u} \in C^0([0,T];\boldsymbol{V})$ *of (2.19). Moreover (2.16) holds and there is a constant* C_1 *depending solely on* $c_{\boldsymbol{V}}$, $\|\alpha\|_{L^\infty(\mathbb{R})}$, $c_\mathcal{K}$, Ω, $\|\boldsymbol{G}\|_{C^0([0,T];(L^2(\Omega))^3)}$, $\|\boldsymbol{g}\|_{C^0([0,T];(L^2(\Gamma_\mathcal{N}))^3)}$, ν, λ, *and* μ, *such that*

$$\|\operatorname{div}\boldsymbol{u}(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 \quad \forall t \in [0,T]. \tag{2.24}$$

Then, our existence and uniqueness result reads as follows

Theorem 2.3. *Under assumptions (2.7)-(2.12), and for* α *fulfilling (2.13) in the precise sense that*

$$C_2 := c_0 - \vartheta_c c_\alpha c_\mathcal{K} C_1 > 0, \tag{2.25}$$

$$\big(\vartheta_c(2\vartheta_c + 1)c_\alpha c_\mathcal{K}\big)^2 \le C_2(\lambda + 2\mu/3), \tag{2.26}$$

Problem **(P)** *has one and only one solution.*

Remark 2.4. Note that, from (1.1) it turns out that the quantity

$$c_0 \;-\; \vartheta\alpha''(\vartheta)\chi_2 \operatorname{div}\boldsymbol{u}$$

(coefficient of ϑ_t in (1.1)) represents the actual specific heat of the shape memory body. In this sense, (2.25) has to be regarded as a non degeneracy condition for the energy balance equation in (1.1). In the same spirit, (2.26) stands for a compatibility condition among the data.

The forthcoming Sections 4 and 5 are devoted to the proof of Theorem **2.3**.

3 Statement of the Scheme and Main Results

Now it is worth introducing our approximation of Problem **(P)**. To this aim, let $\mathcal{P}$ be a partition of the time interval $[0,T]$, namely

$$\mathcal{P} := \{0 = t^0 < t^1 < \cdots < t^{N-1} < t^N = T\}, \tag{3.1}$$

with variable step $\tau^i := t^i - t^{i-1}$. No *a priori* constraints are imposed on the time steps and $\tau := \max_{1\le i\le N} \tau^i$ denotes the diameter of partition $\mathcal{P}$. Let us set

$$F^i := \frac{1}{\tau^i}\int_{t^{i-1}}^{t^i} F(\cdot,t)\,dt \;\in L^2(\Omega), \quad f^i := f(\cdot,t^i) \in L^2(\partial\Omega), \tag{3.2}$$

for $i = 1,\ldots,N$, and

$$\boldsymbol{G}^i := \boldsymbol{G}(\cdot,t^i) \in (L^2(\Omega))^3, \quad \boldsymbol{g}^i := \boldsymbol{g}(\cdot,t^i) \in (L^2(\Gamma_{\mathcal{N}}))^3, \tag{3.3}$$

for $i = 0,1,\ldots,N$. Note that, by virtue of (2.7)-(2.10), definitions (3.2)-(3.3) make sense.

Moreover, we introduce two families of approximating initial data depending on $\mathcal{P}$ and fulfilling

$$\{\vartheta_{0\mathcal{P}}\} \in H^1(\Omega), \quad \big\{[\chi_{1,0\mathcal{P}}, \chi_{2,0\mathcal{P}}]\big\} \in K. \tag{3.4}$$

Now, let $\boldsymbol{u}_{0\mathcal{P}} \in \boldsymbol{V}$ be the related initial displacement (cf. (2.15)), namely the solution of the variational equality

$$a(\boldsymbol{u}_{0\mathcal{P}},\boldsymbol{v}) + \big(\alpha(\vartheta_{0\mathcal{P}})\chi_{2,0\mathcal{P}},\, \operatorname{div}\boldsymbol{v}\big) = \int_\Omega \boldsymbol{G}^0\cdot\boldsymbol{v}\,dx + \int_{\Gamma_{\mathcal{N}}} \boldsymbol{g}^0\cdot\boldsymbol{v}\,d\Gamma \qquad \forall\, \boldsymbol{v}\in\boldsymbol{V}. \tag{3.5}$$

Then, the approximating problem can be stated as follows.

Problem $(\mathbf{P}_{\mathcal{P}})$. Find the vectors $\{\Theta^i\}_{i=0}^N \in (H^1(\Omega))^{N+1}$, $\{\boldsymbol{U}^i\}_{i=0}^N \in \boldsymbol{V}^{N+1}$, $\{\mathcal{X}_j^i\}_{i=0}^N \in (L^2(\Omega))^{N+1}$, for $j=1,2$, fulfilling

$$\Theta^0 = \vartheta_{0\mathcal{P}}, \quad \boldsymbol{U}^0 = \boldsymbol{u}_{0\mathcal{P}}, \quad [\mathcal{X}_1^0, \mathcal{X}_2^0] = [\chi_{1,0\mathcal{P}}, \chi_{2,0\mathcal{P}}], \tag{3.6}$$

and such that the following equations and conditions hold for $i = 1, \ldots, N$,

$$[\mathcal{X}_1^i, \mathcal{X}_2^i] \in K, \tag{3.7}$$

$$\begin{aligned}
&\left(c_0 \frac{\Theta^i - \Theta^{i-1}}{\tau^i} - L \frac{\mathcal{X}_1^i - \mathcal{X}_1^{i-1}}{\tau^i}, \varphi\right) + h \int_\Omega \nabla\Theta^i \cdot \nabla\varphi \, dx + \eta \int_{\partial\Omega} (\Theta^i - f^i)\, \varphi \, d\Gamma \\
&+ \left(\frac{\big(\alpha(\Theta^i) - \Theta^i \alpha'(\Theta^i)\big) \mathcal{X}_2^i \operatorname{div} \boldsymbol{U}^i - \big(\alpha(\Theta^{i-1}) - \Theta^{i-1}\alpha'(\Theta^{i-1})\big)\mathcal{X}_2^{i-1} \operatorname{div} \boldsymbol{U}^{i-1}}{\tau^i}, \varphi \right) \\
&= \big(F^i, \varphi\big) \qquad \forall\, \varphi \in H^1(\Omega),
\end{aligned} \tag{3.8}$$

$$a(\boldsymbol{U}^i, \boldsymbol{v}) + \Big(\alpha(\Theta^i)\mathcal{X}_2^i, \operatorname{div} \boldsymbol{v}\Big) = \int_\Omega \boldsymbol{G}^i \cdot \boldsymbol{v}\, dx + \int_{\Gamma_\mathcal{N}} \boldsymbol{g}^i \cdot \boldsymbol{v}\, d\Gamma \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.9}$$

$$\begin{aligned}
&k \sum_{j=1}^2 \left(\frac{\mathcal{X}_j^i - \mathcal{X}_j^{i-1}}{\tau^i}, \mathcal{X}_j^i - \gamma_j \right) + \ell\Big(\Theta^i - \vartheta^*, \mathcal{X}_1^i - \gamma_1\Big) \\
&\qquad + \Big(\alpha(\Theta^i) \operatorname{div} \boldsymbol{U}^{i-1}, \mathcal{X}_2^i - \gamma_2\Big) \le 0 \quad \forall [\gamma_1, \gamma_2] \in K.
\end{aligned} \tag{3.10}$$

By virtue of (3.5), (3.9) and Lemma **2.2**, it is straightforward to check that the following estimate holds

$$\| \operatorname{div} \boldsymbol{U}^i \|_{L^\infty(\Omega)} \le C_1 \qquad \text{for } i = 0, 1, \ldots, N, \tag{3.11}$$

where C_1 is the same constant of relation (2.24).

Let us stress that the previous scheme is fully implicit in both the energy and the momentum equations. Regarding Problem $(\mathbf{P}_\mathcal{P})$, we have that

Lemma 3.1. *Under assumptions (2.7)-(2.10), (2.12), (2.25)-(2.26), and (3.4), for any partition $\mathcal{P}$, Problem $(\mathbf{P}_\mathcal{P})$ has at least one solution.*

Proof. Thanks to (3.4), it suffices to show that, given a quadruple $(\Theta^{i-1}, \mathcal{X}_1^{i-1}, \mathcal{X}_2^{i-1}, \boldsymbol{U}^{i-1}) \in H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \boldsymbol{V}$, the scheme (3.7)-(3.10) has a solution $(\Theta^{i-1}, \mathcal{X}_1^{i-1}, \mathcal{X}_2^{i-1}, \boldsymbol{U}^{i-1}) \in H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \boldsymbol{V}$, for any value of the time step τ^i. To this aim, we apply Schauder fixed point theorem. As a first step, substitute Θ^i with $\widetilde{\Theta}$ in (3.10) and denote by $[\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2] =: [B_1(\widetilde{\Theta}), B_2(\widetilde{\Theta})]$ the solution to the resulting elementary variational inequality. Next, by replacing in (3.9) the terms Θ^i and $\mathcal{X}_2^i$ with $\widetilde{\Theta}$ and $B_2(\widetilde{\Theta})$, respectively, one may find the unique solution $\widetilde{\boldsymbol{U}} \in \boldsymbol{V}$ to the variational equality. Finally, denoting by $D(\widetilde{\Theta}, \widetilde{\mathcal{X}}_2) := \operatorname{div} \widetilde{\boldsymbol{U}}$, it is straightforward to check that the estimate (2.24) holds for $\operatorname{div} \widetilde{\boldsymbol{U}}$ as well. We deal with (3.8) by substituting $\mathcal{X}_1^i$, $\mathcal{X}_2^i, \operatorname{div} \boldsymbol{U}^i$, and $(\alpha(\Theta^i) - \Theta^i\alpha'(\Theta^i))$ with $\widetilde{\mathcal{X}}_1$, $\widetilde{\mathcal{X}}_2, \operatorname{div} \widetilde{\boldsymbol{U}}$, and $(\alpha(\widetilde{\Theta}) - \widetilde{\Theta}\alpha'(\widetilde{\Theta}))$, respectively. The existence of a unique solution $\Theta =: E(\widetilde{\Theta}, \widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2, \operatorname{div} \widetilde{\boldsymbol{U}})$ is then ensured by the Lax-Milgram lemma. Moreover, by testing (3.8) on Θ with the help of (2.6), (2.12), (2.24), and the elementary inequality (which will be used in the sequel of the paper without any explicit recall) $ab \le (\delta a^2 + b^2/\delta)/2$ for all $a, b \in \mathbb{R}$, $\delta > 0$, it is straightforward to choose a constant C_3 which depends on c_α, $c_\mathcal{K}$, ϑ_c, C_1, $\|f^i\|_{L^2(\partial\Omega)}$, η, h, $\|F^i\|_{L^2(\Omega)}$, T, L, and C_2, such that the following estimate

$$\|\Theta\|^2 + \tau^i \int_\Omega |\nabla\Theta|^2\, dx + \tau^i \int_{\partial\Omega} |\Theta|^2\, d\Gamma \le C_3 \tag{3.12}$$

holds. Thus, by defining

$$S(\widetilde{\Theta}) := E\big(\widetilde{\Theta}, B_1(\widetilde{\Theta}), B_2(\widetilde{\Theta}), D(\widetilde{\Theta}, B_2(\widetilde{\Theta}))\big),$$

it turns out that S maps $L^2(\Omega)$ into a compact and convex subset since we have that the estimate (3.12) is independent of $\widetilde{\Theta}$. In order to apply Schauder fixed point theorem, it remains to show that S is continuous with respect to the topology of $L^2(\Omega)$. Indeed, it suffices to prove the Lipschitz continuity of operators B_1, B_2, D, and E. Regarding B_1, B_2, and D this property has already been proved in [**?**]. Then, we choose two quadruples $(\widetilde{\Theta}, \widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2, \operatorname{div} \widetilde{\boldsymbol{U}})$ and $(\overline{\Theta}, \overline{\mathcal{X}}_1, \overline{\mathcal{X}}_2, \operatorname{div} \overline{\boldsymbol{U}})$. By making use of (2.6) and (2.12), one may easily find a positive constant C_4, depending solely on $L, \vartheta_c, c_\alpha, c_\mathcal{K}, C_1, C_2$, such that

$$\begin{aligned} &\|E(\widetilde{\Theta}, \widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2, \operatorname{div} \widetilde{\boldsymbol{U}}) - E(\overline{\Theta}, \overline{\mathcal{X}}_1, \overline{\mathcal{X}}_2, \operatorname{div} \overline{\boldsymbol{U}})\|^2 \\ &\leq\ C_4\big(\|\widetilde{\Theta} - \overline{\Theta}\|^2 + \|\widetilde{\mathcal{X}}_1 - \overline{\mathcal{X}}_1\|^2 + \|\widetilde{\mathcal{X}}_2 - \overline{\mathcal{X}}_2\|^2 + \|\operatorname{div} \widetilde{\boldsymbol{U}} - \operatorname{div} \overline{\boldsymbol{U}}\|^2\big). \end{aligned}$$

Finally, we conclude for a constant C_5 which depends only on data and fulfills

$$\|S(\widetilde{\Theta}) - S(\overline{\Theta})\|^2\ \leq\ C_5\|\widetilde{\Theta} - \overline{\Theta}\|^2$$

for every $\widetilde{\Theta}, \overline{\Theta} \in L^2(\Omega)$, whence S is continuous and the assertion is proved. ☐

We stress that the forthcoming results of the paper to not rely at all on the uniqueness of a discrete solution. Indeed, both the convergence result and the error estimate hold for any discrete solution as well. Nevertheless, in view of numerical implementation, we prefer to devise here an uniqueness result for Problem $(\mathbf{P}_\mathcal{P})$. Namely, by choosing a partition $\mathcal{P}$ fine enough, we also achieve the following

Lemma 3.2. *Under assumptions (2.7)-(2.10), (2.12), (2.25)-(2.26), (3.4), and for any partition* $\mathcal{P}$ *with diameter* τ *small enough, the solution to Problem* $(\mathbf{P}_\mathcal{P})$ *is unique.*

Proof. We just sketch this argument, since it is very close to other proofs which will be detailed in the sequel of the paper. Let us reason by contradiction assuming that, given a quadruple $(\Theta^{i-1}, \mathcal{X}_1^{i-1}, \mathcal{X}_2^{i-1}, \boldsymbol{U}^{i-1})$, two solutions to (3.7)-(3.10) (at level i) exist. We term the latter two solutions as $(\widetilde{\Theta}, \widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2, \widetilde{\boldsymbol{U}})$ and $(\widehat{\Theta}, \widehat{\mathcal{X}}_1, \widehat{\mathcal{X}}_2, \widehat{\boldsymbol{U}})$, and set

$$\overline{\Theta} = \widetilde{\Theta} - \widehat{\Theta}, \quad \overline{\mathcal{X}}_1 = \widetilde{\mathcal{X}}_1 - \widehat{\mathcal{X}}_1, \quad \overline{\mathcal{X}}_2 = \widetilde{\mathcal{X}}_2 - \widehat{\mathcal{X}}_2, \quad \overline{\boldsymbol{U}} = \widetilde{\boldsymbol{U}} - \widehat{\boldsymbol{U}}.$$

Next, we write relations (3.8) and (3.9) for both the solutions, take the difference and test the resulting equations on $\varphi = \overline{\Theta}$ and $\boldsymbol{v} = \overline{\boldsymbol{U}}$, respectively. Owing to (2.3)-(2.4), (2.6), (2.12), and (2.24)-(2.25), one easily obtains

$$\begin{aligned} &C_2\|\overline{\Theta}\|^2\ +\ \tau^i h \int_\Omega |\nabla \overline{\Theta}|^2 dx\ +\ \tau^i \eta \int_{\partial\Omega} |\overline{\Theta}|^2 d\Gamma \\ &\qquad \leq\ 2\vartheta_c^2 c_\alpha C_1 \|\overline{\mathcal{X}}_2\|\,\|\overline{\Theta}\|\ +\ 2\vartheta_c^2 c_\alpha c_\mathcal{K} \|\operatorname{div} \overline{\boldsymbol{U}}\|\,\|\overline{\Theta}\|, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\frac{c_{\boldsymbol{V}}}{2}\|\overline{\boldsymbol{U}}\|_{\boldsymbol{V}}^2\ +\ \frac{(\lambda + 2\mu/3)}{2}\|\operatorname{div} \overline{\boldsymbol{U}}\|^2\ +\ \frac{\nu}{2}\|\nabla(\operatorname{div} \overline{\boldsymbol{U}})\|_{(L^2(\Omega))^3}^2 \\ &\qquad \leq\ \vartheta_c^2 c_\alpha \|\overline{\mathcal{X}}_2\|\,\|\operatorname{div} \overline{\boldsymbol{U}}\|\ +\ \vartheta_c c_\alpha c_\mathcal{K} \|\overline{\Theta}\|\,\|\operatorname{div} \overline{\boldsymbol{U}}\|. \end{aligned} \tag{3.14}$$

Since relation (2.26) ensures that

$$\vartheta_c c_\alpha c_{\mathcal{K}}(2\vartheta_c+1)\,\|\overline{\Theta}\|\,\|\operatorname{div}\overline{\boldsymbol{U}}\| \;\le\; \frac{3}{4}C_2\|\overline{\Theta}\|^2 \;+\; \frac{(\lambda+2\mu/3)}{3}\|\operatorname{div}\overline{\boldsymbol{U}}\|^2,$$

by taking the sum of inequalities (3.13)-(3.14) we easily infer that

$$\begin{aligned} &\frac{C_2}{4}\|\overline{\Theta}\|^2 \;+\; \frac{c_{\boldsymbol{V}}}{2}\|\overline{\boldsymbol{U}}\|_{\boldsymbol{V}}^2 \;+\; \frac{(\lambda+2\mu/3)}{6}\|\operatorname{div}\overline{\boldsymbol{U}}\|^2 \\ &\quad\le\; 2\vartheta_c^2 c_\alpha C_1\|\overline{\mathcal{X}}_2\|\,\|\overline{\Theta}\| \;+\; \vartheta_c^2 c_\alpha\|\overline{\mathcal{X}}_2\|\,\|\operatorname{div}\overline{\boldsymbol{U}}\| \\ &\quad\le\; \frac{C_2}{8}\|\overline{\Theta}\|^2 \;+\; \frac{(\lambda+2\mu/3)}{12}\|\operatorname{div}\overline{\boldsymbol{U}}\|^2 \;+\; C_6\|\overline{\mathcal{X}}_2\|^2 \end{aligned} \tag{3.15}$$

where

$$C_6 \;:=\; \frac{8(\vartheta_c^2 c_\alpha C_1)^2}{C_2} \;+\; \frac{3(\vartheta_c^2 c_\alpha)^2}{(\lambda+2\mu/3)}.$$

As regards the variational inequality (3.10), arguing as above we infer

$$\frac{k}{\tau^i}\sum_{j=1}^{2}\|\overline{\mathcal{X}}_j\|^2 \;\le\; \ell\,\|\overline{\Theta}\|\,\|\overline{\mathcal{X}}_1\| \;+\; \vartheta_c c_\alpha C_1\|\overline{\Theta}\|\,\|\overline{\mathcal{X}}_2\|,$$

thus, it is straightforward to fix a positive constant, say C_7, which depends on $k, \ell, \vartheta_c, c_\alpha$, and C_1, and fulfills

$$\sum_{j=1}^{2}\|\overline{\mathcal{X}}_j\|^2 \;\le\; \tau^i C_7\|\overline{\Theta}\|^2. \tag{3.16}$$

Finally, looking back to (3.15) and choosing

$$\tau^i \;\le\; \tau \;\le C_2/(16\,C_6 C_7),$$

we conclude that

$$\frac{C_2}{16}\|\overline{\Theta}\|^2 \;+\; \frac{c_{\boldsymbol{V}}}{2}\|\overline{\boldsymbol{U}}\|_{\boldsymbol{V}}^2 \;\le 0.$$

Hence, $\overline{\vartheta}=0$, $\overline{\boldsymbol{U}}=\boldsymbol{0}$ and, recalling (3.16), $\overline{\mathcal{X}}_1=\overline{\mathcal{X}}_2=0$ as well. □

By virtue of Lemma **3.1**, we may fix some convenient notations. Given $\{W^i\}_{i=0}^N$ in the linear space $\mathcal{W}$, set

$$\begin{aligned} &\overline{W}_{\mathcal{P}}(t) = W_{\mathcal{P}}(t) := W^0 \quad \text{ for } t\le 0, \\ &\overline{W}_{\mathcal{P}}(t) := W^i, \quad W_{\mathcal{P}}(t) := W^i + \frac{W^i-W^{i-1}}{\tau^i}(t-t^i) \\ &\text{for } t\in\ (t^{i-1},t^i],\ i=1,\ldots,N. \end{aligned} \tag{3.17}$$

Moreover, we define an operator $\mathcal{T}_{\mathcal{P}}$ related to the partition $\mathcal{P}$. If $\phi:(0,T]$ is a piecewise constant function on $\mathcal{P}$, namely $\phi(t)=\phi^i$ for $t\in\ (t^{i-1},t^i]$, $i=1,\ldots,N$, we set

$$(\mathcal{T}_{\mathcal{P}}\phi)(t) := \phi^{i-1} \quad \text{ for } t\in\ (t^{i-1},t^i],\ i=1,\ldots,N. \tag{3.18}$$

Owing to (3.17) and (3.18) we may conveniently rewrite relations (3.8)-(3.10) as

$$\Big(\partial_t\Big(c_0\Theta_{\mathcal P}+\Big(\big(\alpha(\Theta)-\Theta\alpha'(\Theta)\big)\mathcal X_2\,\mathrm{div}\,\boldsymbol U\Big)_{\mathcal P}\Big),\varphi\Big)$$

$$+\,h\int_\Omega \nabla\overline{\Theta}_{\mathcal P}\cdot\nabla\varphi\,dx \;+\; \eta\int_{\partial\Omega}(\overline{\Theta}_{\mathcal P}-\overline{f}_{\mathcal P})\varphi\,d\Gamma \;=\; \Big(\overline{F}_{\mathcal P}+L\partial_t\mathcal X_{1,\mathcal P},\varphi\Big)$$

$$\forall\varphi\in H^1(\Omega),\quad \text{a.e. in } (0,T), \tag{3.19}$$

$$a(\overline{\boldsymbol U}_{\mathcal P},\boldsymbol v)\;+\;\Big(\alpha(\overline{\Theta}_{\mathcal P})\overline{\mathcal X}_{2,\mathcal P},\,\mathrm{div}\,\boldsymbol v\Big)\;=\;\int_\Omega\overline{\boldsymbol G}_{\mathcal P}\cdot\boldsymbol v\,dx\;+\;\int_{\Gamma_{\mathcal N}}\overline{\boldsymbol g}_{\mathcal P}\cdot\boldsymbol v\,d\Gamma$$

$$\forall\,\boldsymbol v\in\boldsymbol V,\quad \text{a.e. in } (0,T), \tag{3.20}$$

$$k\sum_{j=1}^{2}\Big(\partial_t\mathcal X_{j,\mathcal P},\overline{\mathcal X}_{j,\mathcal P}-\gamma_j\Big)\;+\;\ell\Big(\overline{\Theta}_{\mathcal P}-\vartheta^*,\overline{\mathcal X}_{1,\mathcal P}-\gamma_1\Big)$$

$$+\;\Big(\alpha(\overline{\Theta}_{\mathcal P})\mathcal T_{\mathcal P}\,\mathrm{div}\,\overline{\boldsymbol U}_{\mathcal P},\overline{\mathcal X}_{2,\mathcal P}-\gamma_2\Big)\;\le\;0\quad\forall\,[\gamma_1,\gamma_2]\in K,\quad \text{a.e. in } (0.T). \tag{3.21}$$

The derivation of our error estimates requires additional regularity for the function F. More precisely, we ask

$$F\in BV([0,T];L^2(\Omega)). \tag{3.22}$$

From assumptions (2.8)-(2.10), (3.22), and definitions (3.2)-(3.3) we deduce the existence of a positive constant C_8 such that

$$\|F-\overline{F}_{\mathcal P}\|_{L^1(0,T;L^2(\Omega))}\;+\;\|f-\overline{f}_{\mathcal P}\|_{L^1(0,T;L^2(\partial\Omega))}$$

$$+\;\|\boldsymbol G-\overline{\boldsymbol G}_{\mathcal P}\|_{L^2(0,T;(L^2(\Omega))^3)}\;+\;\|\boldsymbol g-\overline{\boldsymbol g}_{\mathcal P}\|_{L^2(0,T;(L^2(\Gamma_{\mathcal N}))^3)}\;\le\;C_8\,\tau, \tag{3.23}$$

as easy calculations provide. Besides, we choose initial values such that

$$\|\vartheta_0-\vartheta_{0\mathcal P}\|\;+\;\sum_{j=1}^{2}\|\chi_{j,0}-\chi_{j,0\mathcal P}\|\;\le\;C_9\,\tau, \tag{3.24}$$

for some positive constant C_9. Moreover, as a consequence of (3.24), taking the difference between (2.15) and (3.5) and choosing $\boldsymbol v=\boldsymbol u_0-\boldsymbol u_{0\mathcal P}$, relations (2.3)-(2.4), (2.6), (2.24), and (2.14) ensure

$$\|\boldsymbol u_0-\boldsymbol u_{0\mathcal P}\|_{\boldsymbol V}\;+\;\|\,\mathrm{div}\,\boldsymbol u_0-\,\mathrm{div}\,\boldsymbol u_{0\mathcal P}\|\;\le C_{10}\,\tau, \tag{3.25}$$

for a proper constant C_{10}, depending on $\vartheta_c, c_\alpha, C_1, C_9, \lambda, \mu$, and $c_{\boldsymbol V}$.

Now, we state our error estimate.

Theorem 3.3. *Under assumptions (2.8)-(2.10), (2.25)-(2.26), (3.22) and (3.24), let* $(\vartheta,\,\boldsymbol u,\,\chi_1,\,\chi_2)$, $\{\Theta^i,\,\boldsymbol U^i,\,\mathcal X^i,\,\mathcal X^i\}_{i=0}^N$ *be solutions to Problem* **(P)** *and Problem* $(\mathbf P_{\mathcal P})$, *respectively, and* $\Theta_{\mathcal P}$, $\overline{\Theta}_{\mathcal P}$, $\mathcal X_{1,\mathcal P}$, $\mathcal X_{2,\mathcal P}$, $\overline{\boldsymbol U}_{\mathcal P}$, *be as in (3.17). Then, there exists a positive constant* C_{11}, *depending only on data, such that, for every partition* $\mathcal P$ *the following estimate holds*

$$\|\vartheta-\Theta_{\mathcal P}\|_{L^2(0,T;L^2(\Omega))}\;+\;\sup_{t\in[0,T]}\Big\|\int_0^t(\vartheta-\overline{\Theta}_{\mathcal P})(s)ds\Big\|_{H^1(\Omega)}$$

$$+\;\|\boldsymbol u-\boldsymbol U_{\mathcal P}\|_{L^2(0,T;\boldsymbol V)}\;+\;\sum_{j=1}^{2}\|\chi_j-\mathcal X_{j,\mathcal P}\|_{C^0([0,T];L^2(\Omega))}\le\;C_{11}\,\tau. \tag{3.26}$$

Remark 3.4. We point out that the *a priori* estimate (3.26) is *optimal* with respect to the order of convergence, since the backward Euler method to approximate Problem **(P)** is used. Moreover, our estimate is *optimal with respect to the regularity* of the phase variables $\mathcal{X}_1$, $\mathcal{X}_2$ in the sense of [**?**]. Since no *a priori* constraints between consecutive time steps are imposed in our analysis, (3.26) ensures the possibility of implementing a step-by-step choice of time step sizes as shown in [**?**]. However, let us point out that C_{11} depends exponentially on T, as Gronwall lemma is used in the proof of (3.26).

Remark 3.5. Let us stress that the same error estimate still holds if we replace the terms $\|\vartheta - \Theta_{\mathcal{P}}\|_{L^2(0,T;L^2(\Omega))}$ and $\|\boldsymbol{u} - \boldsymbol{U}_{\mathcal{P}}\|_{L^2(0,T;\boldsymbol{V})}$ with $\|\vartheta - \overline{\Theta}_{\mathcal{P}}\|_{L^2(0,T;L^2(\Omega))}$ and $\|\boldsymbol{u} - \overline{\boldsymbol{U}}_{\mathcal{P}}\|_{L^2(0,T;\boldsymbol{V})}$, respectively (see the following Lemma **4.1**).

4 Existence

In this section we prove the existence result of Theorem **2.3**. This proof follows closely the argument devised in [**?**], so that it will be just sketched, referring to that paper for the details.

First of all, we establish some estimates for the approximating solutions which are independent of $\mathcal{P}$. More precisely, one finds two constants τ^* and C_{12}, which depend on $\|\alpha\|_{L^\infty(\mathbb{R})}$, Ω, $\Gamma_{\mathcal{N}}$, $c_{\boldsymbol{V}}$, L, ϑ_c, c_α, C_1, C_2, and T, such that, for every partition $\mathcal{P}$ with diameter $\tau < \tau^*$, one has (see [**?**, Lemma 3.1 and eq. (4.27)])

$$\begin{aligned} &\|\Theta_{\mathcal{P}}\|_{H^1(0,T;L^2(\Omega))} + \|\overline{\Theta}_{\mathcal{P}}\|_{L^\infty(0,T;H^1(\Omega))} + \|\boldsymbol{U}_{\mathcal{P}}\|_{H^1(0,T;\boldsymbol{V})} + \|\operatorname{div}\boldsymbol{U}_{\mathcal{P}}\|_{L^\infty(Q)} \\ &\quad + \|\operatorname{div}\boldsymbol{U}_{\mathcal{P}}\|_{H^1(0,T;H^1(\Omega))} + \sum_{j=1}^{2}\|\mathcal{X}_{j,\mathcal{P}}\|_{H^1(0,T;L^2(\Omega))\,\cap\,L^\infty(Q)} \le C_{12}. \end{aligned} \tag{4.1}$$

Indeed, relations (2.12) and (4.1) ensures also that

$$\big\|\big(\alpha(\Theta) - \Theta\alpha'(\Theta)\big)_{\mathcal{P}}\big\|_{W^{1,\infty}(Q)} \quad \text{is bounded independently of } \mathcal{P}. \tag{4.2}$$

For the sake of convenience, we collect here some convergence results which will be useful in the sequel.

Lemma 4.1. *Let $\Theta_{\mathcal{P}}$, $\overline{\Theta}_{\mathcal{P}}$, $\boldsymbol{U}_{\mathcal{P}}$, $\overline{\boldsymbol{U}}_{\mathcal{P}}$, $\mathcal{X}_{j,\mathcal{P}}$, $\overline{\mathcal{X}}_{j,\mathcal{P}}$ for $j = 1,2$ be defined as in (3.17) and fulfill (4.1). Moreover, let $\mathcal{T}_{\mathcal{P}}$ be defined in (3.18). Then we have*

$$\|\Theta_{\mathcal{P}} - \overline{\Theta}_{\mathcal{P}}\|_{L^2(Q)} \le C\tau, \tag{4.3}$$

$$\|\Theta_{\mathcal{P}} - \overline{\Theta}_{\mathcal{P}}\|_{L^\infty(0,T;L^2(\Omega))} \le C\sqrt{\tau}, \tag{4.4}$$

$$\|\boldsymbol{U}_{\mathcal{P}} - \overline{\boldsymbol{U}}_{\mathcal{P}}\|_{L^2(0,T;\boldsymbol{V})} \le C\tau, \tag{4.5}$$

$$\|\boldsymbol{U}_{\mathcal{P}} - \overline{\boldsymbol{U}}_{\mathcal{P}}\|_{L^\infty(0,T;\boldsymbol{V})} \le C\sqrt{\tau}, \tag{4.6}$$

$$\|\operatorname{div}\boldsymbol{U}_{\mathcal{P}} - \mathcal{T}_{\mathcal{P}}\operatorname{div}\overline{\boldsymbol{U}}_{\mathcal{P}}\|_{L^2(0,T;H^1(\Omega))} \le C\tau, \tag{4.7}$$

$$\|\mathcal{X}_{j,\mathcal{P}} - \overline{\mathcal{X}}_{j,\mathcal{P}}\|_{L^2(Q)} \le C\tau \quad \textit{for } j = 1,2. \tag{4.8}$$

Proof. Note that the proofs of (4.3)-(4.6), and (4.8) follow easily from (4.1). Let us just check (4.7). We have that

$$\begin{aligned} &\|\operatorname{div} \boldsymbol{U}_{\mathcal{P}} - \mathcal{T}_{\mathcal{P}} \operatorname{div} \overline{\boldsymbol{U}}_{\mathcal{P}}\|^2_{L^2(0,T;H^1(\Omega))} \\ = &\sum_{i=1}^{N} \int_{t^{i-1}}^{t^i} \left\| \operatorname{div} U^{i-1} - \Big(\alpha_i(t) \operatorname{div} U^i + \big(1-\alpha_i(t)\big) \operatorname{div} U^{i-1}\Big) \right\|^2_{H^1(\Omega)} dt \end{aligned} \qquad (4.9)$$

where $\alpha_i(t) = (t - t^{i-1})/\tau^i$ for $t \in [t^{i-1}, t^i]$, $i = 1, \ldots, N$. Thus, due to (4.1), one has

$$\|\operatorname{div} \boldsymbol{U}_{\mathcal{P}} - \mathcal{T}_{\mathcal{P}} \operatorname{div} \overline{\boldsymbol{U}}_{\mathcal{P}}\|^2_{L^2(0,T;H^1(\Omega))} \;\le\; \sum_{i=1}^{N} \frac{\tau^i}{3} \| \operatorname{div} U^i - \operatorname{div} U^{i-1}\|^2_{H^1(\Omega)} \;\le\; C\tau^2.$$

□

By taking the limit in equations (3.19)-(3.21) as the diameter of partitions tends to 0, one shows that Problem **(P)** has at least one solution. Indeed, the estimates (4.1)-(4.2) and well-known compactness results (see, for instance, [**?**, Cor. 4]) ensure that there exist $\vartheta, \boldsymbol{u}$, and ψ such that, possibly taking subsequences (not relabeled),

$$\begin{aligned} \Theta_{\mathcal{P}} &\longrightarrow \vartheta && \text{weakly star in } H^1(0,T;L^2(\Omega)) \text{ and} \\ & && \text{strongly in } C^0([0,T];L^2(\Omega)), && (4.10) \\ \boldsymbol{U}_{\mathcal{P}} &\longrightarrow \boldsymbol{u} && \text{weakly in } H^1(0,T;\boldsymbol{V}), && (4.11) \\ \operatorname{div} \boldsymbol{U}_{\mathcal{P}} &\longrightarrow \operatorname{div} \boldsymbol{u} && \text{strongly in } C^0([0,T];L^2(\Omega)), && (4.12) \\ \big(\alpha(\Theta) - \Theta\alpha'(\Theta)\big)_{\mathcal{P}} &\longrightarrow \psi && \text{weakly star in } W^{1,\infty}(Q), && (4.13) \end{aligned}$$

as the diameter τ tends to 0 (clearly much more is true). Moreover, the previous convergences, along with Lemma **4.1**, entail that [**?**, Sect. 5]

$$\mathcal{X}_{1,\mathcal{P}} \text{ and } \mathcal{X}_{2,\mathcal{P}} \text{ are Cauchy sequences in } C^0([0,T];L^2(\Omega)) \qquad (4.14)$$

and we obviously deduce from relation (2.12) that

$$\alpha(\Theta) - \Theta\alpha'(\Theta) \longrightarrow \alpha(\vartheta) - \vartheta\alpha'(\vartheta) \quad \text{strongly in } C^0([0,T];L^2(\Omega)). \qquad (4.15)$$

It remains to prove that $\vartheta, \chi_1, \chi_2$, and $\operatorname{div} \boldsymbol{u}$ fulfill (2.18). To this aim, note that easy calculations yield

$$\begin{aligned} \partial_t\Big(\big(\alpha(\Theta) - \Theta\alpha'(\Theta)\big)\mathcal{X}_2 \operatorname{div} \boldsymbol{U}\Big)_{\mathcal{P}} \;=\; &\partial_t\big(\alpha(\Theta) - \Theta\alpha'(\Theta)\big)_{\mathcal{P}} \overline{\mathcal{X}}_{2,\mathcal{P}} \operatorname{div} \overline{\boldsymbol{U}}_{\mathcal{P}} \\ +\mathcal{T}_{\mathcal{P}}\big(\alpha(\overline{\Theta}_{\mathcal{P}}) - \overline{\Theta}_{\mathcal{P}}\alpha'(\overline{\Theta}_{\mathcal{P}})\big)\, \partial_t \mathcal{X}_{2,\mathcal{P}} \operatorname{div} \overline{\boldsymbol{U}}_{\mathcal{P}} &+ \mathcal{T}_{\mathcal{P}}\Big(\big(\alpha(\overline{\Theta}_{\mathcal{P}}) - \overline{\Theta}_{\mathcal{P}}\alpha'(\overline{\Theta}_{\mathcal{P}})\big)\mathcal{X}_{2,\mathcal{P}}\Big)\, \partial_t(\operatorname{div} \boldsymbol{U}_{\mathcal{P}}). \end{aligned}$$

Referring to [**?**], we only have to deal with the first term in the right hand side above since the passage to the limit in the other two terms is ensured by the above listed convergences. In particular, let us prove the following useful lemma

Lemma 4.2. *Let E and F be normed linear spaces. Moreover, let $g : E \to F$ be a Lipschitz continuous function of Lipschitz constant L_g, $\{u^i\}_{i=0}^N \in E^{N+1}$, and $\big(g(u)\big)_{\mathcal{P}}$ and $u_{\mathcal{P}}$ be defined as in (3.17). Then, we have that*

$$\big\|\big(g(u)\big)_{\mathcal{P}} - g(u_{\mathcal{P}})\big\|_{L^2(0,T;F)} \;\le\; \sqrt{2/15}\, L_g\, \tau \|\partial_t u_{\mathcal{P}}\|_{L^2(0,T;E)}. \tag{4.16}$$

Proof. Fix $t \in (t^{i-1}, t^i]$ for $i = 1, \ldots, N$, and let $\alpha_i(t) = (t - t^{i-1})/\tau^i$; we have that

$$\begin{aligned}
&\big\|\big(g(u)\big)_{\mathcal{P}}(t) - g(u_{\mathcal{P}}(t))\big\|_F \\
&= \big\|\alpha_i(t) g(u^i) + \big(1-\alpha_i(t)\big) g(u^{i-1}) - g\big(\alpha_i(t) u^i + \big(1-\alpha_i(t)\big)u^{i-1}\big)\big\|_F \\
&= \Big\|\alpha_i(t) g(u^i) + \big(1-\alpha_i(t)\big) g(u^{i-1}) - \alpha_i(t) g\big(\alpha_i(t) u^i + \big(1-\alpha_i(t)\big)u^{i-1}\big) \\
&- \big(1-\alpha_i(t)\big) g\big(\alpha_i(t) u^i + \big(1-\alpha_i(t)\big)u^{i-1}\big)\Big\|_F \\
&\le 2L_g\, \alpha_i(t)\big(1-\alpha_i(t)\big)\|u^i - u^{i-1}\|_E,
\end{aligned}$$

since we have that both $\alpha_i(t)$ and $\big(1-\alpha_i(t)\big)$ are nonnegative. Owing to the previous inequality and taking into account easy calculations, we have

$$\begin{aligned}
\big\|\big(g(u)\big)_{\mathcal{P}} - g(u_{\mathcal{P}})\big\|^2_{L^2(0,T;F)} &= \int_0^t \big\|\big(g(u)\big)_{\mathcal{P}}(t) - g(u_{\mathcal{P}}(t))\big\|_F^2\, dt \\
&\le 4L_g^2 \sum_{i=1}^N \left(\left(\int_{t^{i-1}}^{t^i} \alpha_i^2(t)\big(1-\alpha_i(t)\big)^2\, dt\right) \|u^i - u^{i-1}\|_E^2\right) \\
&\le \frac{2}{15} L_g^2 \sum_{i=1}^N \tau^i \|u^i - u^{i-1}\|_E^2 \;=\; \frac{2}{15} L_g^2 \tau^2 \|\partial_t u_{\mathcal{P}}\|^2_{L^2(0,T;E)}
\end{aligned}$$

whence the assertion follows. ☐

An application of the previous result (along with (2.12) and (4.1)) ensures that

$$\big(\alpha(\Theta) - \Theta\alpha'(\Theta)\big)_{\mathcal{P}} - \big(\alpha(\Theta_{\mathcal{P}}) - \Theta_{\mathcal{P}}\alpha'(\Theta_{\mathcal{P}})\big) \longrightarrow 0 \qquad \text{strongly in } L^2(Q)$$

so that, owing to (2.12), (4.10), (4.13), and (4.15), we have that $\psi = \alpha(\vartheta) - \vartheta\alpha'(\vartheta)$, whence, recalling (4.13), one in particular infers that

$$\partial_t\big(\alpha(\Theta) - \Theta\alpha'(\Theta)\big)_{\mathcal{P}} \longrightarrow \partial_t\big(\alpha(\vartheta) - \vartheta\alpha'(\vartheta)\big) \qquad \text{weakly in } L^2(Q).$$

Then, owing to the latter convergence and arguing as in [**?**], one easily checks that relation (2.18) is fulfilled and the proof is complete.

5 Uniqueness

The following proof follows closely the argument investigated in [**?**]. In this respect, we just suggest how to proceed, and omit most of the computations. We reason by contradiction. Let $(\vartheta^1, \chi_1^1, \chi_2^1, \boldsymbol{u}^1)$ and $(\vartheta^2, \chi_1^2, \chi_2^2, \boldsymbol{u}^2)$ be two solutions to Problem **(P)** and set $\overline{\vartheta} := \vartheta^1 - \vartheta^2$, $\overline{\chi}_1 := \chi_1^1 - \chi_1^2$, $\overline{\chi}_2 := \chi_2^1 - \chi_2^2$, $\overline{\boldsymbol{u}} := \boldsymbol{u}^1 - \boldsymbol{u}^2$. Let us take the difference between equation (2.18) written for $(\vartheta^1, \chi_1^1, \chi_2^1, \boldsymbol{u}^1)$ and the same equation for $(\vartheta^2, \chi_1^2, \chi_2^2, \boldsymbol{u}21)$, integrate the resulting relation on $(0,t)$, choose $\varphi = \overline{\vartheta}(t)$, and integrate once more over $(0,t)$. Owing to relations (2.6), (2.12), (2.14), (2.24), Hölder inequality, and the mean value theorem, one infers that

$$\frac{11}{12} C_2 \|\overline{\vartheta}\|^2_{L^2(0,t;L^2(\Omega))} + \frac{h}{2} \left\| \nabla \left(\int_0^t \overline{\vartheta}(s) ds \right) \right\|^2_{(L^2(\Omega))^3} + \frac{\eta}{2} \left\| \int_0^t \overline{\vartheta}(s) ds \right\|^2_{L^2(\partial\Omega)}$$
$$\leq 2\vartheta_c^2 c_\alpha c_{\mathcal{K}} \int_0^t \| \operatorname{div} \overline{\boldsymbol{u}}(s)\| \, \|\overline{\vartheta}(s)\| ds + C_{13} \int_0^t \sum_{j=1}^2 \|\overline{\chi}_j(s)\|^2 ds, \tag{5.1}$$

where the constant C_{13} depends on ϑ_c, c_α, L, C_1, and , C_2. Next, we write relation (2.20) for $(\vartheta^1, \chi_1^1, \chi_2^1, \boldsymbol{u}^1)$ (letting $[\gamma_1, \gamma_2] = [\chi_1^2(t), \chi_2^2(t)]$) and $(\vartheta^2, \chi_1^2, \chi_2^2, \boldsymbol{u}^2)$ (letting $[\gamma_1, \gamma_2] = [\chi_1^1(t), \chi_2^1(t)]$, respectively). Taking the sum of the two inequalities, integrating in time, and owing to relations (2.24) and (2.14), one easily finds a proper constant C_{14} depending on ϑ_c, c_α, λ, μ, ℓ, C_1, and , C_2, such that

$$\frac{k}{2} \sum_{j=1}^2 \|\overline{\chi}_j(t)\|^2 \leq \frac{C_2}{12} \|\overline{\vartheta}\|^2_{L^2(0,t;L^2(\Omega))}$$
$$+ \frac{(\lambda + 2\mu/3)}{24} \| \operatorname{div} \overline{\boldsymbol{u}}\|^2_{L^2(0,t;L^2(\Omega))} + C_{14} \int_0^t \sum_{j=1}^2 \|\overline{\chi}_j(s)\|^2 ds. \tag{5.2}$$

Finally, we write (2.19) for both $(\vartheta^1, \chi_1^1, \chi_2^1, \boldsymbol{u}^1)$ and $(\vartheta^2, \chi_1^2, \chi_2^2, \boldsymbol{u}^2)$ take the difference between the two resulting equalities, choose $\boldsymbol{v} = \overline{\boldsymbol{u}}$, and integrate in time. Owing to (2.3)-(2.4), (2.17), and (2.14) one has

$$\frac{c_{\boldsymbol{V}}}{2} \|\overline{\boldsymbol{u}}\|^2_{L^2(0,t;\boldsymbol{V})} + \frac{\nu}{2} \|\nabla(\operatorname{div} \overline{\boldsymbol{u}})\|^2_{L^2(0,t;(L^2(\Omega))^3)} + \frac{11(\lambda + 2\mu/3)}{24} \| \operatorname{div} \overline{\boldsymbol{u}}\|^2_{L^2(0,t;L^2(\Omega))}$$
$$\leq \vartheta_c c_\alpha c_{\mathcal{K}} \int_0^t \|\overline{\vartheta}(s)\| \, \| \operatorname{div} \overline{\boldsymbol{u}}(s)\| \, ds + C_{15} \int_0^t \|\overline{\chi}_2(s)\|^2 ds, \tag{5.3}$$

where C_{15} properly depends on ϑ_c, c_α, λ, μ . Now, we take the sum between (5.1), (5.2), and (5.3). Since (2.26) ensures that

$$\Big(\vartheta_c (2\vartheta_c + 1) c_\alpha c_{\mathcal{K}} \Big) \int_0^t \|\overline{\vartheta}(s)\| \, \| \operatorname{div} \overline{\boldsymbol{u}}(s)\| \, ds$$
$$\leq \frac{3}{4} C_2 \|\overline{\vartheta}\|^2_{L^2(0,t;L^2(\Omega))} + \frac{(\lambda + 2\mu/3)}{3} \| \operatorname{div} \overline{\boldsymbol{u}}\|^2_{L^2(0,t;L^2(\Omega))},$$

one infers, for all $t \in (0,T)$,

$$\frac{C_2}{12} \|\overline{\vartheta}\|^2_{L^2(0,t;L^2(\Omega))} + \frac{h}{2} \left\| \nabla \left(\int_0^t \overline{\vartheta}(s) \, ds \right) \right\|^2_{(L^2(\Omega))^3} + \frac{\eta}{2} \left\| \int_0^t \overline{\vartheta}(s) \, ds \right\|^2_{L^2(\partial\Omega)}$$
$$+ \frac{c_{\boldsymbol{V}}}{2} \|\overline{\boldsymbol{u}}\|^2_{L^2(0,t;\boldsymbol{V})} + \frac{\nu}{2} \|\nabla(\operatorname{div} \overline{\boldsymbol{u}})\|^2_{L^2(0,t;(L^2(\Omega))^3)} + \frac{(\lambda + 2\mu/3)}{12} \| \operatorname{div} \overline{\boldsymbol{u}}\|^2_{L^2(0,t;L^2(\Omega))}$$
$$+ \frac{k}{2} \sum_{j=1}^2 \|\overline{\chi}_j(t)\|^2 \leq C_{16} \int_0^t \sum_{j=1}^2 \|\overline{\chi}_j(s)\|^2 ds,$$

where $C_{16} := 3\max\{C_{13}, C_{14}, C_{15}\}$. Hence, applying Gronwall lemma (see, e.g., the version reported in [**?**, Thm. 1]) it follows that the solution to Problem **(P)** is unique.

6 Error Estimates

Henceforth, C stands for a positive constant depending eventually on data but independent of $\mathcal{P}$. Of course, C may vary from line to line. Moreover, in the sequel of the paper, where no confusion arises, we will drop the subscript $\mathcal{P}$ from functions $\Theta_{\mathcal{P}}$, $\overline{\Theta}_{\mathcal{P}}$, $\mathcal{X}_{1,\mathcal{P}}$, $\mathcal{X}_{2,\mathcal{P}}$, $\overline{\mathcal{X}}_{1,\mathcal{P}}$, $\overline{\mathcal{X}}_{2,\mathcal{P}}$, $\boldsymbol{U}_{\mathcal{P}}$, $\overline{\boldsymbol{U}}_{\mathcal{P}}$, $\overline{F}_{\mathcal{P}}$, $\overline{f}_{\mathcal{P}}$, $\overline{\boldsymbol{G}}_{\mathcal{P}}$, and $\overline{\boldsymbol{g}}_{\mathcal{P}}$.

Let us start up by handling the variational inequalities (2.20) and (3.21). To this end, we refer the reader to [**?**, **?**] where this analysis is devised in an abstract setting, and to [**?**, **?**] where it has been applied. We choose $[\gamma_1,\gamma_2] = [\overline{\mathcal{X}}_1(t),\overline{\mathcal{X}}_2(t)]$ in (2.20), $[\gamma_1,\gamma_2] = [\chi_1(t),\chi_2(t)]$ in (3.21) and sum the corresponding two inequalities. By means of easy calculations one infers

$$
\begin{aligned}
&k\sum_{j=1}^{2}\Big(\partial_t(\chi_j-\mathcal{X}_j),(\chi_j-\mathcal{X}_j)\Big) \;+\; k\sum_{j=1}^{2}\Big(\partial_t\mathcal{X}_j,(\overline{\mathcal{X}}_j-\mathcal{X}_j)\Big)\\
&+\ell\Big(\vartheta-\overline{\Theta},\chi_1-\overline{\mathcal{X}}_1\Big) \;+\; \Big(\alpha(\vartheta)\operatorname{div}\boldsymbol{u}-\alpha(\overline{\Theta})\mathcal{T}_{\mathcal{P}}\operatorname{div}\overline{\boldsymbol{U}},\chi_2-\overline{\mathcal{X}}_2\Big) \;\le\; 0.
\end{aligned}
$$

Taking the integral over $(0,t)$, we have

$$
\frac{k}{2}\sum_{j=1}^{2}\|(\chi_j-\mathcal{X}_j)(t)\|^2 \;=\; \sum_{i=1}^{5}I_i(t), \tag{6.1}
$$

for all $t\in(0,T)$, where

$$
\begin{aligned}
I_1 &= \frac{k}{2}\sum_{j=1}^{2}\|\chi_{j,0}-\chi_{j,0\mathcal{P}}\|^2,\\
I_2(t) &= k\int_0^t\sum_{j=1}^{2}\Big(\partial_t\mathcal{X}_j(s),(\mathcal{X}_j-\overline{\mathcal{X}}_j)(s)\Big)\,ds,\\
I_3(t) &= -\ell\int_0^t(\vartheta-\overline{\Theta},\chi_1-\overline{\mathcal{X}}_1)(s)\,ds,\\
I_4(t) &= -\int_0^t\Big(\big(\alpha(\vartheta)\operatorname{div}\boldsymbol{u}-\alpha(\overline{\Theta})\operatorname{div}\overline{\boldsymbol{U}}\big)(s),(\chi_2-\overline{\mathcal{X}}_2)(s)\Big)\,ds,\\
I_5(t) &= -\int_0^t\Big(\alpha(\overline{\Theta})(\operatorname{div}\overline{\boldsymbol{U}}-\mathcal{T}_{\mathcal{P}}\operatorname{div}\overline{\boldsymbol{U}})(s),(\chi_2-\overline{\mathcal{X}}_2)(s)\Big)ds.
\end{aligned}
$$

Clearly, position (3.24) ensures that

$$
I_1 \;\le\; C\tau^2. \tag{6.2}
$$

Our next aim is to control the *residual* quantity $I_2(t)$. Let $t \in (t^{i-1}, t^i]$, for some $i = 1, \dots, N$. We have that

$$\sum_{j=1}^{2} \big(\partial_t \mathcal{X}_j, \mathcal{X}_j - \overline{\mathcal{X}}_j\big)(t) = \sum_{j=1}^{2} \left(\frac{\mathcal{X}_j^i - \mathcal{X}_j^{i-1}}{\tau^i}, \alpha_i(t)\mathcal{X}_j^i + \big(1-\alpha_i(t)\big)\mathcal{X}_j^{i-1} - \mathcal{X}_j^i \right)$$
$$= \big(\alpha_i(t) - 1\big)\tau^i \sum_{j=1}^{2} \left\| \frac{\mathcal{X}_j^i - \mathcal{X}_j^{i-1}}{\tau^i} \right\|^2,$$

where, once again, $\alpha_i(t) = (t - t^{i-1})/\tau^i$ (note that $|\alpha_i| \leq 1$). Then, one infers

$$\sum_{j=1}^{2} \big(\partial_t \mathcal{X}_j, \mathcal{X}_j - \overline{\mathcal{X}}_j\big)(t) \leq 0 \qquad \forall t \in (0, T),$$

and, consequently,

$$I_2(t) \leq 0 \qquad \forall t \in (0, T). \tag{6.3}$$

Regarding $I_3(t)$ and $I_4(t)$, by virtue of (2.14), (4.3), and (4.8), one infers

$$I_3(t) \leq \frac{C_2}{52} \|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} + C \left(\int_0^t \|(\chi_1 - \mathcal{X}_1)(s)\|^2 ds + \tau^2 \right), \tag{6.4}$$

$$\begin{aligned} I_4(t) &\leq \int_0^t \left| \Big(\alpha(\vartheta)(\operatorname{div} \boldsymbol{u} - \operatorname{div} \overline{\boldsymbol{U}}), \chi_2 - \overline{\mathcal{X}}_2 \Big) \right| (s)\, ds \\ &+ \int_0^t \left| \Big(\operatorname{div} \overline{\boldsymbol{U}} (\alpha(\vartheta) - \alpha(\overline{\Theta})), \chi_2 - \overline{\mathcal{X}}_2 \Big) \right| (s)\, ds \\ &\leq \frac{(\lambda + 2\mu/3)}{24} \| \operatorname{div} \boldsymbol{u} - \operatorname{div} \overline{\boldsymbol{U}} \|^2_{L^2(0,t;L^2(\Omega))} + \frac{C_2}{52} \|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} \\ &+ C \left(\int_0^t \|(\chi_2 - \mathcal{X}_2)(s)\|^2 ds + \tau^2 \right). \end{aligned} \tag{6.5}$$

Please note that the constant C_2 in the calculations above is exactly the one appearing in (2.25). Moreover, let us stress that the choice of the quantity $C_2/52$ and $(\lambda + 2\mu/3)/24$, although it is not straightforward at the moment, is strictly related with assumptions (2.25) and (2.4), respectively, as will be clear in the sequel. Due to relations (2.14) and (4.7)-(4.8), it is possible to control $I_5(t)$ as follows

$$\begin{aligned} I_5(t) &\leq \frac{(\vartheta_c^2 c_\alpha)^2}{2} \| \operatorname{div} \overline{\boldsymbol{U}} - \mathcal{T}_{\mathcal{P}} \operatorname{div} \overline{\boldsymbol{U}} \|^2_{L^2(0,t;L^2(\Omega))} + \frac{1}{2} \int_0^t \|(\chi_2 - \overline{\mathcal{X}}_2)(s)\|^2 ds \\ &\leq C \left(\int_0^t \|(\chi_2 - \mathcal{X}_2)(s)\|^2 ds + \tau^2 \right). \end{aligned} \tag{6.6}$$

In order to get a control of the function $\vartheta, -\Theta_{\mathcal{P}}$ with respect to the norm of $L^2(0,T;L^2(\Omega))$ we consider the integral of (2.18) and (3.19) over $(0,t)$ for $t \in (0,T)$, and obtain, respectively,

$$
\begin{aligned}
&c_0\Big(\big(\vartheta(t)-\vartheta_0\big),\varphi\Big) \;+\; h\int_\Omega \nabla\left(\int_0^t \vartheta(s)\,ds\right)\cdot\nabla\varphi\,dx \;+\; \eta\int_{\partial\Omega}\left(\int_0^t (\vartheta-f)(s)\,ds\right)\varphi\,d\Gamma \\
&= \int_0^t \Big(F(s),\varphi\Big)\,ds \;+\; L\Big(\big(\chi_1(t)-\chi_{1,0}\big),\varphi\Big) \\
&+ \Big(\Big(\vartheta\alpha'(\vartheta)-\alpha(\vartheta)\Big)\chi_2\,\mathrm{div}\,\boldsymbol{u}(t),\varphi\Big) \;-\; \Big(\Big(\vartheta_0\alpha'(\vartheta_0)-\alpha(\vartheta_0)\Big)\chi_{2,0}\,\mathrm{div}\,\boldsymbol{u}_0,\varphi\Big), \\
&\qquad\qquad \forall\varphi\in H^1(\Omega),
\end{aligned}
\tag{6.7}
$$

$$
\begin{aligned}
&c_0\Big(\big(\Theta(t)-\vartheta_{0\mathcal{P}}\big),\varphi\Big) \;+\; h\int_\Omega \nabla\left(\int_0^t \overline{\Theta}(s)\,ds\right)\cdot\nabla\varphi\,dx \\
&+\; \eta\int_{\partial\Omega}\left(\int_0^t (\overline{\Theta}-\overline{f})(s)\,ds\right)\varphi\,d\Gamma \;=\; \int_0^t \Big(\overline{F}(s),\varphi\Big)\,ds \\
&+\; L\Big(\big(\mathcal{X}_1(t)-\chi_{1,0\mathcal{P}}\big),\varphi\Big) \;+\; \Big(\Big(\Theta\alpha'(\Theta)-\alpha(\Theta)\Big)\mathcal{X}_2\,\mathrm{div}\,\boldsymbol{U}(t),\varphi\Big) \\
&-\; \Big(\Big(\vartheta_{0\mathcal{P}}\alpha'(\vartheta_{0\mathcal{P}})-\alpha(\vartheta_{0\mathcal{P}})\Big)\chi_{2,0\mathcal{P}}\,\mathrm{div}\,\boldsymbol{u}_{0\mathcal{P}},\varphi\Big) \;+\; \Big(\mathcal{R}(t),\varphi\Big), \\
&\qquad\qquad \forall\varphi\in H^1(\Omega),
\end{aligned}
\tag{6.8}
$$

where the *residual* term $\mathcal{R}(t)$ is defined by

$$
\mathcal{R}(t) := \int_0^t \Big(\Big(\big(\Theta\alpha'(\Theta)-\alpha(\Theta)\big)\mathcal{X}_2\,\mathrm{div}\,\boldsymbol{U}\Big)_{\mathcal{P}} - \big(\Theta_{\mathcal{P}}\alpha'(\Theta_{\mathcal{P}})-\alpha(\Theta_{\mathcal{P}})\big)\mathcal{X}_{2,\mathcal{P}}\,\mathrm{div}\,\boldsymbol{U}_{\mathcal{P}}\Big).
$$

Taking the difference between (6.7) and (6.8), choosing $\varphi=(\vartheta-\overline{\Theta})(t)$ and integrating over $(0,t)$, one infers

$$
\begin{aligned}
&c_0\|\vartheta-\Theta\|^2_{L^2(0,t;L^2(\Omega))} \;+\; \frac{h}{2}\left\|\nabla\left(\int_0^t(\vartheta-\overline{\Theta})(s)ds\right)\right\|^2_{(L^2(\Omega))^3} \\
&\qquad +\frac{\eta}{2}\left\|\int_0^t(\vartheta-\overline{\Theta})(s)ds\right\|^2_{L^2(\partial\Omega)} \;=\; \sum_{i=6}^{13} I_i(t),
\end{aligned}
\tag{6.9}
$$

for all $t \in (0,T)$, where

$$
\begin{aligned}
I_6(t) &= \int_0^t \Big(c_0(\vartheta_0 - \vartheta_{0\mathcal{P}}) - L(\chi_{1,0} - \chi_{1,0\mathcal{P}}), (\vartheta - \overline{\Theta})(s)\Big)ds,\\
I_7(t) &= \int_0^t \Big(c_0(\vartheta - \Theta)(s), (\overline{\Theta} - \Theta)(s)\Big)ds,\\
I_8(t) &= \int_0^t \Big(L(\chi_1 - \mathcal{X}_1)(s), (\vartheta - \overline{\Theta})(s)\Big)ds,\\
I_9(t) &= \int_0^t \left(\int_0^s (F - \overline{F})(r)dr, (\vartheta - \overline{\Theta})(s)ds\right),\\
I_{10}(t) &= \eta \int_0^t \int_{\partial\Omega} \left(\int_0^s (f - \overline{f})(r)dr\right) (\vartheta - \overline{\Theta})(s)d\Gamma\, ds,\\
I_{11}(t) &= -\int_0^t \Big(\Big(\vartheta_0\alpha'(\vartheta_0) - \alpha(\vartheta_0)\Big)\chi_{2,0}\,\mathrm{div}\,\boldsymbol{u}_0\\
&\qquad - \Big(\vartheta_{0\mathcal{P}}\alpha'(\vartheta_{0\mathcal{P}}) - \alpha(\vartheta_{0\mathcal{P}})\Big)\chi_{2,0\mathcal{P}}\,\mathrm{div}\,\boldsymbol{u}_{0\mathcal{P}}, (\vartheta - \overline{\Theta})(s)\Big)ds,
\end{aligned}
$$

$$
\begin{aligned}
I_{12}(t) &= \int_0^t \Big(\Big(\vartheta\alpha'(\vartheta) - \alpha(\vartheta)\Big)\chi_2\,\mathrm{div}\,\boldsymbol{u}(s) - \Big(\Theta\alpha'(\Theta) - \alpha(\Theta)\Big)\mathcal{X}_2\,\mathrm{div}\,\boldsymbol{U}(s),\\
&\qquad (\vartheta - \overline{\Theta})(s)\Big)ds,
\end{aligned}
$$

$$
I_{13}(t) = \int_0^t \Big(\mathcal{R}(s), (\vartheta - \overline{\Theta})(s)\Big)ds,
$$

Our next aim is to control the right hand side of (6.9). To this end, due to (3.24) and (4.3), we handle $I_6(t)$, $I_7(t)$, and $I_8(t)$ as follows

$$
\begin{aligned}
I_6(t) &\leq \frac{C_2}{52}\|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))}\\
&+ C\Big(\|\vartheta_0 - \vartheta_{0\mathcal{P}}\|^2 + \|\chi_{1,0} - \chi_{1,0\mathcal{P}}\|^2 + \tau^2\Big)\\
&\leq \frac{C_2}{52}\|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2, \qquad (6.10)\\
I_7(t) &\leq \frac{C_2}{52}\|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2, \qquad (6.11)\\
I_8(t) &\leq \frac{C_2}{52}\|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} + C\left(\int_0^t \|(\chi_1 - \mathcal{X}_1)(s)\|^2 ds + \tau^2\right). \qquad (6.12)
\end{aligned}
$$

Next, we control $I_9(t)$ by virtue of (3.23) and (4.3) as

$$
\begin{aligned}
I_9(t) &\leq \frac{C_2}{52}\|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))}\\
&+ C\left(\|F - \overline{F}\|^2_{L^1(0,T;L^2(\Omega))} + \tau^2\right)\\
&\leq \frac{C_2}{52}\|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2. \qquad (6.13)
\end{aligned}
$$

Regarding $I_{10}(t)$, relation (3.23) and an integration by parts yield

$$\begin{aligned} I_{10}(t) &\leq \eta \left| \int_{\partial\Omega} \left(\int_0^t (f - \overline{f})(s) ds \right) \left(\int_0^t (\vartheta - \overline{\Theta})(s) ds \right) d\Gamma \right| \\ &+ \eta \left| \int_0^t \int_{\partial\Omega} (f - \overline{f})(s) \left(\int_0^s (\vartheta - \overline{\Theta})(r) dr \right) d\Gamma \, ds \right| \\ &\leq \frac{\eta}{4} \left\| \int_0^t (\vartheta - \overline{\Theta})(s) ds \right\|^2_{L^2(\partial\Omega)} + C \, \| f - \overline{f} \|^2_{L^1(0,T;L^2(\partial\Omega))} \\ &+ \eta \int_0^t \| (f - \overline{f})(s) \|_{L^2(\partial\Omega)} \left\| \int_0^s (\vartheta - \overline{\Theta})(r) dr \right\|_{L^2(\partial\Omega)} ds. \end{aligned} \tag{6.14}$$

In order to bound $I_{11}(t)$, we reason as follows

$$I_{11}(t) = I_{14}(t) + I_{15}(t) + I_{16}(t)$$

where

$$\begin{aligned} I_{14}(t) &= -\int_0^t \Big(\big((\vartheta_0 \alpha'(\vartheta_0) - \alpha(\vartheta_0)) - (\vartheta_{0\mathcal{P}} \alpha'(\vartheta_{0\mathcal{P}}) - \alpha(\vartheta_{0\mathcal{P}})) \big) \chi_{2,0} \operatorname{div} \boldsymbol{u}_0, \\ &\qquad\qquad (\vartheta - \overline{\Theta})(s) \Big) ds, \\ I_{15}(t) &= -\int_0^t \Big(\big(\vartheta_{0\mathcal{P}} \alpha'(\vartheta_{0\mathcal{P}}) - \alpha(\vartheta_{0\mathcal{P}}) \big) (\chi_{2,0} - \chi_{2,0\mathcal{P}}) \operatorname{div} \boldsymbol{u}_0, (\vartheta - \overline{\Theta})(s) \Big) ds, \\ I_{16}(t) &= -\int_0^t \Big(\big(\vartheta_{0\mathcal{P}} \alpha'(\vartheta_{0\mathcal{P}}) - \alpha(\vartheta_{0\mathcal{P}}) \big) \chi_{2,0\mathcal{P}} (\operatorname{div} \boldsymbol{u}_0 - \operatorname{div} \boldsymbol{u}_{0\mathcal{P}}), (\vartheta - \overline{\Theta})(s) \Big) ds. \end{aligned}$$

Hence, owing to (2.6), (2.14), (2.24), (3.11), (3.24)-(3.25), and (4.3), one obtains

$$\begin{aligned} I_{14}(t) &\leq \vartheta_c c_\alpha c_{\mathcal{K}} C_1 \int_0^t \| \vartheta_0 - \vartheta_{0\mathcal{P}} \| \, \| (\vartheta - \overline{\Theta})(s) \| \, ds \\ &\leq \frac{C_2}{52} \| \vartheta - \Theta \|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2, \end{aligned} \tag{6.15}$$

$$\begin{aligned} I_{15}(t) &\leq 2\vartheta_c^2 c_\alpha C_1 \int_0^t \| \chi_{2,0} - \chi_{2,0\mathcal{P}} \| \, \| (\vartheta - \overline{\Theta})(s) \| \, ds \\ &\leq \frac{C_2}{52} \| \vartheta - \Theta \|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2, \end{aligned} \tag{6.16}$$

$$\begin{aligned} I_{16}(t) &\leq 2\vartheta_c^2 c_\alpha c_{\mathcal{K}} \int_0^t \| \operatorname{div} \boldsymbol{u}_0 - \operatorname{div} \boldsymbol{u}_{0\mathcal{P}} \| \, \| (\vartheta - \overline{\Theta})(s) \| \, ds \\ &\leq \frac{C_2}{52} \| \vartheta - \Theta \|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2. \end{aligned} \tag{6.17}$$

Thus, collecting (6.15)-(6.17), we have

$$I_{11}(t) \leq \frac{3C_2}{52} \| \vartheta - \Theta \|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2. \tag{6.18}$$

The same analysis exploited for $I_{11}(t)$ applies to $I_{12}(t)$ as well. For instance consider

$$I_{12}(t) \;=\; I_{17}(t) \;+\; I_{18}(t) \;+\; I_{19}(t)$$

where

$$\begin{aligned} I_{17}(t) &= \int_0^t \Big(\Big(\big(\vartheta\alpha'(\vartheta) - \alpha(\vartheta)\big) - \big(\Theta\alpha'(\Theta) - \alpha(\Theta)\big)\Big) \chi_2 \operatorname{div} \boldsymbol{u}(s), (\vartheta - \overline{\Theta})(s) \Big) ds, \\ I_{18}(t) &= \int_0^t \Big(\big(\Theta\alpha'(\Theta) - \alpha(\Theta)\big)(\chi_2 - \mathcal{X}_2) \operatorname{div} \boldsymbol{u}(s), (\vartheta - \overline{\Theta})(s) \Big) ds, \\ I_{19}(t) &= \int_0^t \Big(\big(\Theta\alpha'(\Theta) - \alpha(\Theta)\big) \mathcal{X}_2 (\operatorname{div} \boldsymbol{u} - \operatorname{div} \boldsymbol{U})(s), (\vartheta - \overline{\Theta})(s) \Big) ds. \end{aligned}$$

By virtue of relations (2.6), (2.24), (2.14), and (4.3), one infers

$$\begin{aligned} I_{17}(t) &\le \vartheta_c c_\alpha c_{\mathcal{K}} C_1 \int_0^t \|(\vartheta - \Theta)(s)\| \, \|(\vartheta - \overline{\Theta})(s)\| \, ds \\ &= \vartheta_c c_\alpha c_{\mathcal{K}} C_1 \left(\|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} + \int_0^t \|(\vartheta - \Theta)(s)\| \, \|(\Theta - \overline{\Theta})(s)\| \, ds \right) \\ &\le \left(\vartheta_c c_\alpha c_{\mathcal{K}} C_1 \;+\; \frac{C_2}{52} \right) \|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} \;+\; C\tau^2, \qquad (6.19) \\ I_{18}(t) &\le 2\vartheta_c^2 c_\alpha C_1 \int_0^t \|(\chi_2 - \mathcal{X}_2)(s)\| \, \|(\vartheta - \overline{\Theta})(s)\| \, ds \\ &\le \frac{C_2}{52} \|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} \;+\; C \left(\int_0^t \|(\chi_2 - \mathcal{X}_2)(s)\|^2 ds \;+\; \tau^2 \right), \qquad (6.20) \\ I_{19}(t) &\le 2\vartheta_c^2 c_\alpha c_{\mathcal{K}} \int_0^t \|(\operatorname{div} \boldsymbol{u} - \operatorname{div} \boldsymbol{U})(s)\| \, \|(\vartheta - \overline{\Theta})(s)\| \, ds. \qquad (6.21) \end{aligned}$$

The reader should notice that the term $\vartheta_c c_\alpha c_{\mathcal{K}} C_1 \|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))}$ in (6.19) is to be handled by means of the non-degeneracy assumption (2.25). On the other hand, let us stress that the term $I_{19}(t)$ will be controlled jointly with the forthcoming term $I_{20}(t)$ by making a crucial use of (2.26). Moreover, according to (6.19)-(6.21), we conclude for

$$\begin{aligned} I_{12}(t) \;&\le\; \left(\vartheta_c c_\alpha c_{\mathcal{K}} C_1 \;+\; \frac{C_2}{26} \right) \|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} \\ &\quad +\; 2\vartheta_c^2 c_\alpha c_{\mathcal{K}} \int_0^t \|(\operatorname{div} \boldsymbol{u} - \operatorname{div} \boldsymbol{U})(s)\| \, \|(\vartheta - \overline{\Theta})(s)\| \, ds \\ &\quad +\; C \left(\int_0^t \|(\chi_2 - \mathcal{X}_2)(s)\|^2 ds \;+\; \tau^2 \right). \qquad (6.22) \end{aligned}$$

Finally, we deal with the residual term $I_{13}(t)$. It is straightforward to obtain that

$$I_{13}(t) \le \frac{C_2}{52} \|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))}$$

$$+ C \left(\left\| \Big(\big(\Theta\alpha'(\Theta) - \alpha(\Theta)\big) \mathcal{X}_2 \operatorname{div} \boldsymbol{U} \Big)_{\mathcal{P}} - \big(\Theta_{\mathcal{P}}\alpha'(\Theta_{\mathcal{P}}) - \alpha(\Theta_{\mathcal{P}})\big) \mathcal{X}_{2,\mathcal{P}} \operatorname{div} \boldsymbol{U}_{\mathcal{P}} \right\|^2_{L^2(Q)} + \tau^2 \right)$$

and, recalling (2.6), (2.12), and (2.24), an application of Lemma **4.2** with the choice $E = (L^2(\Omega))^3$ and $F = L^2(\Omega)$, yields

$$I_{13}(t) \leq \frac{C_2}{52}\|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2. \tag{6.23}$$

Regarding the displacements, let us consider the difference between (2.19) and (3.20) and set $\boldsymbol{v} = (\boldsymbol{u} - \overline{\boldsymbol{U}})(t)$. One has

$$\begin{aligned} &a\Big(\boldsymbol{u} - \overline{\boldsymbol{U}}, \boldsymbol{u} - \overline{\boldsymbol{U}}\Big) \;+\; \Big(\alpha(\vartheta)\chi_2 - \alpha(\overline{\Theta})\overline{\mathcal{X}}_2,\, \operatorname{div}\boldsymbol{u} - \operatorname{div}\overline{\boldsymbol{U}}\Big) \\ =& \int_\Omega (\boldsymbol{G} - \overline{\boldsymbol{G}})\cdot(\boldsymbol{u} - \overline{\boldsymbol{U}})\,dx \;+\; \int_{\Gamma_{\mathcal{N}}} (\boldsymbol{g} - \overline{\boldsymbol{g}})\cdot(\boldsymbol{u} - \overline{\boldsymbol{U}})\,d\Gamma \quad \text{a.e. in } (0,T). \end{aligned}$$

Now, we take the integral over $(0,t)$ for $t \in (0,T)$. Since we have (2.3)-(2.4), it is straightforward to deduce that

$$\begin{aligned} &\frac{c_{\boldsymbol{V}}}{2}\|(\boldsymbol{u} - \overline{\boldsymbol{U}})(t)\|^2_{L^2(0,t;\boldsymbol{V})} \\ &+ \frac{(\lambda + 2\mu/3)}{2}\|(\operatorname{div}\boldsymbol{u} - \operatorname{div}\overline{\boldsymbol{U}})(t)\|^2_{L^2(0,t;L^2(\Omega))} \;\leq\; \sum_{i=20}^{23} I_i(t), \end{aligned} \tag{6.24}$$

where

$$\begin{aligned} I_{20}(t) &= -\int_0^t \Big((\alpha(\vartheta) - \alpha(\overline{\Theta}))\chi_2(s), (\operatorname{div}\boldsymbol{u} - \operatorname{div}\overline{\boldsymbol{U}})(s)\Big)\,ds, \\ I_{21}(t) &= -\int_0^t \Big(\alpha(\overline{\Theta})(\chi_2 - \overline{\mathcal{X}}_2)(s), (\operatorname{div}\boldsymbol{u} - \operatorname{div}\overline{\boldsymbol{U}})(s)\Big)\,ds, \\ I_{22}(t) &= \int_0^t\int_\Omega (\boldsymbol{G} - \overline{\boldsymbol{G}})\cdot(\boldsymbol{u} - \overline{\boldsymbol{U}})\,dx\,ds, \\ I_{23}(t) &= \int_0^t\int_{\Gamma_{\mathcal{N}}} (\boldsymbol{g} - \overline{\boldsymbol{g}})\cdot(\boldsymbol{u} - \overline{\boldsymbol{U}})\,d\Gamma\,ds. \end{aligned}$$

The previous terms may be controlled with the help of (2.6), (2.14), (3.23), and (4.3) as follows

$$I_{20}(t) \leq \vartheta_c c_\alpha c_{\mathcal{K}} \int_0^t \|(\vartheta - \overline{\Theta})(s)\| \, \|(\operatorname{div} \boldsymbol{u} - \operatorname{div} \overline{\boldsymbol{U}})(s)\| \, ds, \tag{6.25}$$

$$\begin{aligned} I_{21}(t) \leq\ & \frac{(\lambda + 2\mu/3)}{24} \|\operatorname{div} \boldsymbol{u} - \operatorname{div} \overline{\boldsymbol{U}}\|^2_{L^2(0,t;L^2(\Omega))} \\ &+ C\Big(\int_0^t \|(\chi_2 - \mathcal{X}_2)(s)\|^2 ds + \tau^2 \Big), \end{aligned} \tag{6.26}$$

$$\begin{aligned} I_{22}(t) &\leq \frac{c_{\boldsymbol{V}}}{8} \|\boldsymbol{u} - \overline{\boldsymbol{U}}\|^2_{L^2(0,t;\boldsymbol{V})} + C \, \|\boldsymbol{G} - \overline{\boldsymbol{G}}\|^2_{L^2(0,T;(L^2(\Omega))^3)} \\ &\leq \frac{c_{\boldsymbol{V}}}{8} \|\boldsymbol{u} - \overline{\boldsymbol{U}}\|^2_{L^2(0,t;\boldsymbol{V})} + C\tau^2, \end{aligned} \tag{6.27}$$

$$\begin{aligned} I_{23}(t) &\leq \frac{c_{\boldsymbol{V}}}{8C_*^2} \|\boldsymbol{u} - \overline{\boldsymbol{U}}\|^2_{L^2(0,t;(L^2(\Gamma_{\mathcal{N}}))^3)} + C \, \|\boldsymbol{g} - \overline{\boldsymbol{g}}\|^2_{L^2(0,T;(L^2(\Gamma_{\mathcal{N}}))^3)} \\ &\leq \frac{c_{\boldsymbol{V}}}{8} \|\boldsymbol{u} - \overline{\boldsymbol{U}}\|^2_{L^2(0,t;\boldsymbol{V})} + C \, \|\boldsymbol{g} - \overline{\boldsymbol{g}}\|^2_{L^2(0,T;(L^2(\Gamma_{\mathcal{N}}))^3)} \\ &\leq \frac{c_{\boldsymbol{V}}}{8} \|\boldsymbol{u} - \overline{\boldsymbol{U}}\|^2_{L^2(0,t;\boldsymbol{V})} + C\tau^2, \end{aligned} \tag{6.28}$$

where the constant C_* stands for the norm of the trace operator from $\boldsymbol{V}$ to $(L^2(\Gamma_{\mathcal{N}}))^3$. Once again we choose the arbitrary constants in the right hand side of relations (6.26)-(6.28) in order to fit the forthcoming analysis.

Next, we take the sum between (6.1), (6.9), and (6.24). At this point the role of assumption (2.26) becomes clear since the latter assumption and (4.3) ensure that

$$\begin{aligned} &\Big(\vartheta_c(2\vartheta_c + 1) c_\alpha c_{\mathcal{K}}\Big) \int_0^t \|(\vartheta - \overline{\Theta})(s)\| \, \|(\operatorname{div} \boldsymbol{u} - \operatorname{div} \overline{\boldsymbol{U}})(s)\| \, ds \\ &\leq \frac{3C_2}{4} \|\vartheta - \overline{\Theta}\|^2_{L^2(0,t;L^2(\Omega))} + \frac{(\lambda + 2\mu/3)}{3} \|\operatorname{div} \boldsymbol{u} - \operatorname{div} \overline{\boldsymbol{U}}\|^2_{L^2(0,t;L^2(\Omega))} \\ &\leq \frac{3C_2}{4} \|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} + \frac{(\lambda + 2\mu/3)}{3} \|\operatorname{div} \boldsymbol{u} - \operatorname{div} \overline{\boldsymbol{U}}\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2. \end{aligned}$$

A first consequence of the previous inequality we have that the sum $I_{19}(t) + I_{20}(t)$ is controlled by the right hand side above (provided that the constant C is properly modified).

Thus, by virtue of (2.25) and taking into account (6.2)-(6.6), (6.10)-(6.14), (6.18), (6.22)-(6.23), and (6.25)-(6.28), one obtains for all $t \in (0,T)$.

$$
\begin{aligned}
&\frac{C_2}{52}\|\vartheta - \Theta\|^2_{L^2(0,t;L^2(\Omega))} + \frac{h}{2}\left\|\nabla\left(\int_0^t (\vartheta - \overline{\Theta})(s)\,ds\right)\right\|^2_{(L^2(\Omega))^3} \\
&\quad + \frac{\eta}{4}\left\|\int_0^t (\vartheta - \overline{\Theta})(s)\,ds\right\|^2_{L^2(\partial\Omega)} + \frac{c_{\boldsymbol{V}}}{4}\|\boldsymbol{u} - \overline{\boldsymbol{U}}\|^2_{L^2(0,t;\boldsymbol{V})} \\
&+ \frac{(\lambda + 2\mu/3)}{12}\|\operatorname{div}\boldsymbol{u} - \operatorname{div}\overline{\boldsymbol{U}}\|^2_{L^2(0,t;L^2(\Omega))} + \frac{k}{2}\sum_{j=1}^{2}\|(\chi_j - \mathcal{X}_j)(t)\|^2 \\
&\quad \leq C\Bigg(\int_0^t \|(f - \overline{f})(s)\|_{L^2(\partial\Omega)}\left\|\int_0^s (\vartheta - \overline{\Theta})(r)\,dr\right\|_{L^2(\partial\Omega)} \\
&\qquad \sum_{j=1}^{2}\int_0^t \|(\chi_j - \mathcal{X}_j)(s)\|^2 ds + \tau^2\Bigg).
\end{aligned}
$$

Finally, an application of Gronwall lemma along with relations (3.23) and (4.5), concludes the proof of Theorem **3.3**.

Remark 6.1. Let us briefly comment the technical motivation for neglecting the term $\alpha(\vartheta)\chi_2\partial_t(\operatorname{div}\boldsymbol{u})$ in (1.7). The latter motivation is connected with the former paper [**?**]. In this contribution the author deals with the uniqueness of a solution to the full three-dimensional Frémond model by reasoning by contradiction. The presence of the nonlinear term $\alpha(\vartheta)\chi_2\partial_t(\operatorname{div}\boldsymbol{u})$ forces the author to establish a *local in time* Gronwall type estimate. Thus, the uniqueness of a solution is proved in the time interval $[0, T^*)$ for a suitably small time $T^* < T$ and the argument is iterated to ensure uniqueness on the whole interval $[0,T)$. Unfortunately, the latter local in time procedure is not adequate for the purpose of the error analysis and we need to establish a Gronwall estimate up to the reference time T. In this respect (see also [**?**]), it turns out to be possible to prove such a *global in time* Gronwall estimate by neglecting the term $\alpha(\vartheta)\chi_2\partial_t(\operatorname{div}\boldsymbol{u})$ in the full energy balance equation (1.7).