Rate-independent evolutions and material modeling

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Preface

Rate-independence is the attribute of those evolution systems whose response is invariant under time rescalings. This feature appears quite ubiquitously with respect to real physical and engineering situations. In particular, the quasi-static evolution of mechanical systems experiencing plasticity, friction, damage, cracks, or various phase transformations often can be modelled by rate-independent systems. The interest in the consideration of rate-independent evolutionary problems has steadily increased since the flourishing of the mathematical theory of hysteresis in the late 70s. Recently, we assisted to a renewed surge of attention for rate-independent evolutions, especially in connection with the description of the thermo or electro-mechanical or magneto-mechanical behavior of materials.

We had the pleasure of organizing an invited special session on

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within the international conference EQUADIFF which took place in Vienna in August 2007. For that occasion we luckily succeeded in bringing together a nice group of experts and young colleagues in the mathematical treatment of rate-independent systems with the aim of sharing results and visions on the future of the field. Most of the participants to our session later submitted a short contribution to the main conference organizers for some collective proceeding publication. However, for rather technical reasons, these conference proceedings have never been published.

We are convinced that the contributions of our invited speakers are still highly interesting and, although partly outdated now because of the rapidly advancing of this pretty active research field, are definitely deserving attention. Hence, in agreement with the organizer of EQUADIFF 2007 and having now eventually confirmed that the mentioned proceedings will not be published, we decided to take action and gathered the contributions from our special session in this volume essentially in their original form submitted in 2007/8 with only minimal editorial changes in particular cases. In our opinion, this serves at least as some recollection (by now, to some extent, historical) of a definite standpoint for the *rate-independent* community, from which all our Authors took the chance to step forward towards new and interesting results.

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On a Neumann parabolic problem with hysteresis: the 3D-case

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Abstract

A parabolic equation in three space variables with a Preisach hysteresis operator and with homogeneous Neumann boundary conditions is shown to admit a unique global regular solution; this can be used to prove the asymptotic stabilization of the solution. The results of this paper improve the content of Ref. [5], where the regularity of the solution was obtained under appropriate smallness conditions on the initial data.

Keywords: Partial differential equations, hysteresis, 3D-case.

1 Introduction

In the paper Ref. [5] we presented a qualitative study on the long time asymptotic stabilization of the solution to the equation

$$\frac{\partial}{\partial t}(f(u) + \mathcal{W}[\lambda, u]) - \triangle u \; = \; 0 \tag{1}$$

with given initial conditions, where $\triangle$ is the Laplace operator on a bounded regular domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ with homogeneous Neumann boundary conditions, f is an increasing function on an interval $[-R, R]$, and $\mathcal{W}[\lambda, \cdot]$ is a Preisach operator with initial memory configuration λ.

In particular in the 3D-case the regularity of the solution was shown under appropriate smallness conditions on the initial data.

The aim of this paper is to show that, by means of a more refined technique, the regularity of the solution (and therefore also the asymptotic stabilization as in Ref. [5]) holds for any choice of the initial data, without any further restrictions. The key point relies on the fact that, in order to obtain the asymptotic stabilization, it is not needed to show the regularity of the solution in the whole space-time cylinder $\Omega \times \mathbb{R}_+$, but only for sufficiently large times. These results therefore improve the corresponding ones in Ref. [5].

The original motivation for this problem comes from soil hydrology. For more details on the derivation of the model equation (1) we refer again to Ref. [5]; for a more detailed discussion on further modeling issues, see for instance Refs. [1], [2], [6], [8].

2 Statement of the problem

In a bounded domain $\Omega \subset \mathbb{R}^3$, with a Lipschitzian boundary and in the time interval $\mathbb{R}_+ = [0, \infty[$ we consider the evolution problem

$$\frac{\partial}{\partial t}(f(u) + w) - \triangle u \;=\; 0\,, \tag{2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given function, $\triangle$ is the Laplace operator, and

$$w(x,t) \;=\; \mathcal{W}[\lambda, u](x,t) \;=\; \int_0^\infty g(r, \wp_r[\lambda(x,\cdot), u(x,\cdot)](t))\, dr$$

is the output of a Preisach operator $\mathcal{W}$ with initial memory configuration λ and generating function $g(r,v) = \int_0^v \varphi(r,z)\, dz$, where φ is a given non-negative function. For a detailed discussion on the Preisach operator we refer to the monographs Refs. [4], [11], [12]; we refer also to more recent results Refs. [9], [10] concerning the alternative one-parametric formulation of the Preisach model based on variational inequalities which is used here.

Equation (2) is coupled with the homogeneous Neumann boundary condition $\frac{\partial u}{\partial n} \;=\; 0$ on $\partial\Omega\times\,]0,\infty[$ and with the initial condition

$$u(x,0) = u_0(x),\;\; w(x,0) = w_0(x) := \int_0^\infty g(r, P[\lambda(x,\cdot), u_0(x)](r))\, dr, \tag{3}$$

where P is defined as $P[\lambda, v](r) \;=\; \max\{v - r, \min\{v + r, \lambda(r)\}\}\,.$

Hyphothesis 2.1. We fix a constant $R > 0$ and assume that

(i) $f \in W^{2,\infty}(-R,R)$, and there exist $f_1 > f_0 > 0$, $f_2 > 0$ such that $|f''(v)| \le f_2$ a.e., $f_0 \le f'(v) \le f_1$ for all $v \in [-R,R]$.

(ii) φ restricted to $Q(R) :=]0,R[\times]-R,R[$ is measurable, and there exist $\beta \in L^1(0,R)$ and $\varphi_1 > 0$ such that, a.e. in $Q(R)$

$$0 \;\le\; \varphi(r,z) \;\le\; \varphi_1\,, \qquad \left|\frac{\partial\varphi}{\partial z}(r,z)\right| \;\le\; \beta(r).$$

(iii) The initial memory configuration λ is a strongly measurable mapping from Ω to $\Lambda_R := \{\lambda \in W^{1,\infty}(0,\infty)\,;\, |\lambda'(r)| \le 1 \text{ a.e.}, \lambda(r) = 0 \text{ for } r \ge R\}$ (endowed with the sup-norm).

(iv) The initial condition u_0 belongs to $W^{2,2}(\Omega)$, and there exist constants $-R < u_* < u^* < R$ such that $u_* \;\le\; u_0(x) \;\le\; u^* \quad \forall x \in \Omega\,.$

We now state the main result of the paper.

Theorem 2.1. *Let Hypothesis* 2.1 *hold. Then Problem* (2) – (3) *admits a unique continuous solution* u *on* $\Omega \times \mathbb{R}_+$ *such that* $u_* \le u(x,t) \le u^*$ *for all* $(x,t) \in \Omega \times \mathbb{R}_+$ *and there exists* $T^* > 0$ *such that* u *has the regularity*

$$\begin{array}{l} \partial_t u, \triangle u \in L^2(\Omega\times]0,\infty[),\;\; \partial_t \nabla u_{|\Omega\times]T^*,\infty[} \in L^2(\Omega\times]T^*,\infty[) \\ \partial_t u, \triangle u_{|\Omega\times]T^*,\infty[} \in L^2(\Omega\times]T^*,\infty[) \cap L^\infty(T^*,\infty; L^2(\Omega)). \end{array} \tag{4}$$

Moreover there exists $u_\infty \in \mathbb{R}$ *such that* $\lim_{t\to\infty} \sup_{x\in\Omega} |u(x,t) - u_\infty| \; = \; 0.$

Remark 2.2. We will confine ourselves in showing in details how to get (4). The asymptotic stabilization then follows as in Refs. [5], Section 4.

3 Proof of Theorem 2.1

We fix a time step $\tau > 0$ and define in Ω for $k \in \mathbb{N}$ a recurrent system

$$\begin{aligned}
&\frac{1}{\tau}(f(u_k) + w_k - f(u_{k-1}) - w_{k-1}) - \triangle u_k \; = \; 0, \qquad (5)\\
&w_k(x) \; = \; \int_0^\infty g(r, \lambda_k(x,r))\, dr\,, \qquad \frac{\partial u_k}{\partial n} \; = \; 0 \quad \text{on} \;\; \partial\Omega,\\
&\lambda_0(x,r) \; := \; P[\lambda(x,\cdot), u_0(x)](r)\,, \quad \lambda_k(x,r) \; := \; P[\lambda_{k-1}(x,\cdot), u_k(x)](r).
\end{aligned}$$

By the argument of Ref. [5], Section 3, this problem admits a unique solution $u_k \in W^{2,2}(\Omega)$ with $u_* \le u_k(x) \le u^*$ for all $x \in \Omega$. We can rewrite (5) as

$$\frac{1}{\tau}(V_k(x) - V_{k-1}(x)) - \triangle u_k(x) \; = \; 0, \qquad (6)$$

where we set $V_k(x) := f(u_k(x)) + w_k(x)$. Using Hypothesis 2.1, we have

$$\begin{aligned}
f_0\,(u_k(x) - u_{k-1}(x))^2 \; &\le \; (V_k(x) - V_{k-1}(x))\,(u_k(x) - u_{k-1}(x)) \qquad (7)\\
&\le \; (f_1 + 2\,\varphi_1\,R)\,(u_k(x) - u_{k-1}(x))^2. \qquad (8)
\end{aligned}$$

Now, we test (6) by $u_k(x) - u_{k-1}(x)$; using (7) and the monotonicity of f, g, and $P[\lambda, \cdot]$, we obtain for every $n \in \mathbb{N}$

$$\frac{f_0}{\tau}\sum_{k=1}^{n}\int_\Omega |u_k(x) - u_{k-1}(x)|^2\; dx + \frac{1}{2}\int_\Omega |\nabla u_n(x)|^2\; dx \; \le \; \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2\; dx\,.$$

On the other hand, this inequality gives, using (6) and (8)

$$\tau\sum_{k=1}^{n}\int_\Omega |\triangle u_k(x)|^2\, dx \le \frac{(f_1 + 2\,\varphi_1\,R)^2}{2\,f_0}\int_\Omega |\nabla u_0(x)|^2\, dx.$$

Therefore there exists a constant $C^* > 0$ such that

$$\tau\sum_{k=1}^{\infty}\int_\Omega (|\nabla u_k(x)|^2 + |\triangle u_k(x)|^2)\, dx < \; C^*. \qquad (9)$$

From now on we set $W_k \; := \; \dfrac{1}{2\tau^2}\displaystyle\int_\Omega (V_{k+1}(x) - V_k(x))(u_{k+1}(x) - u_k(x))\, dx.$ Using a discrete second order energy inequality, we derived in Ref. [5], formula (3.33), the following inequality, valid for $k \ge 1$,

$$\begin{aligned}
&W_k - W_{k-1} + \frac{1}{\tau}\int_\Omega |\nabla(u_{k+1}(x) - u_k(x))|^2\, dx\\
\le\;& \frac{3\gamma}{2\tau^2}\int_\Omega (u_{k+1}(x) - u_k(x))^2 |u_k(x) - u_{k-1}(x)|\, dx, \qquad (10)
\end{aligned}$$

with $\gamma > 0$ depending only on the data of the problem. With the notation of Hypothesis 1.2 of Ref. [5], set

$$\delta := \frac{8\,f_0^2}{81\,\gamma^2\,\mu_4^4(\Omega)}, \qquad T^* = \frac{C^*}{\delta} + 1, \qquad n^* = \left[\frac{T^*}{\tau}\right], \tag{11}$$

where $\mu_4(\Omega)$ is the constant coming from the Gagliardo-Nirenberg inequality. By (9) there exists $k^* \le n^*$ such that

$$\int_\Omega (|\nabla u_{k^*}(x)|^2 + |\triangle u_{k^*}(x)|^2)\,dx < \delta. \tag{12}$$

We test (6) by $(u_k(x) - u_{k-1}(x))$ and sum for $k = k^*+1, \dots, n+1$, getting

$$\sum_{k=k^*}^{n} \tau\,W_k + \frac{1}{4}\int_\Omega |\nabla u_{n+1}(x)|^2\,dx \le \frac{1}{4}\int_\Omega |\nabla u_{k^*}(x)|^2\,dx. \tag{13}$$

Now test (6), corresponding to $k = k^*+1$, by $(u_{k^*+1}(x) - u_{k^*}(x))$; we have

$$W_{k^*} + \frac{1}{\tau}\int_\Omega |\nabla(u_{k^*+1}(x) - u_{k^*}(x))|^2\,dx \le \frac{1}{2f_0}\int_\Omega |\triangle u_{k^*}(x)|^2 dx. \tag{14}$$

At this point, using Hölder's inequality and (7), we may rewrite (10) as

$$\begin{aligned} & W_k - W_{k-1} + \frac{1}{\tau}\int_\Omega |\nabla(u_{k+1}(x) - u_k(x))|^2\,dx \\ \le\ & \frac{3\gamma}{\tau\sqrt{2f_0}}\,W_{k-1}^{1/2}\left(\int_\Omega |u_{k+1}(x) - u_k(x)|^4\,dx\right)^{1/2}, \end{aligned} \tag{15}$$

for $k \ge k^*$. Using now the Gagliardo-Nirenberg inequality (see for example Refs. [3], [7]) and the generalized Young inequality, we get the estimate

$$\begin{aligned} & \left(\int_\Omega |u_{k+1}(x) - u_k(x)|^4\,dx\right)^{1/2} \le 2\mu_4^2(\Omega)\tau^2 \\ & \times\left(\frac{2}{f_0}W_k + \left(\frac{2}{f_0}W_k\right)^{1/4}\left(\frac{1}{\tau^2}\int_\Omega |\nabla(u_{k+1}(x) - u_k(x))|^2\,dx\right)^{3/4}\right). \end{aligned} \tag{16}$$

Combining (15) and (16) we deduce

$$\begin{aligned} & W_k - W_{k-1} + \frac{1}{2\tau}\int_\Omega |\nabla(u_{k+1}(x) - u_k(x))|^2\,dx \\ \le\ & \frac{1}{3f_0}\left(2 + \delta_f^2 W_{k-1}\right)^2 \tau W_k, \end{aligned} \tag{17}$$

where we set $\delta_f := \dfrac{9\,\gamma\,\mu_4^2(\Omega)}{\sqrt{2f_0}}$. For $s > 0$ set

$$h(s) = \frac{1}{(2+\delta_f^2 s)^2}, \quad H(s) = \int_0^s h(\sigma)\,d\sigma = \frac{s}{2(2+\delta_f^2 s)}. \tag{18}$$

Since H is increasing and concave, we have for all $k \in \mathbb{N}$ that

$$(W_k - W_{k-1})\, h(W_{k-1}) \;\geq\; H(W_k) - H(W_{k-1})\,.$$

Multiplying (17) by $h(W_{k-1})$, summing up this time for $k = k^* + 1, \ldots, n$, and using (13) yields

$$\begin{aligned} &H(W_n) - H(W_{k^*}) + \sum_{k=k^*+1}^{n} \frac{h(W_{k-1})}{2\tau} \int_\Omega \left|\nabla(u_{k+1}(x) - u_k(x))\right|^2 dx \\ \leq\;& \frac{1}{3f_0} \sum_{k=k^*+1}^{n} \tau W_k \leq \frac{1}{3f_0} \sum_{k=k^*}^{n} \tau W_k \overset{(13)}{\leq} \frac{1}{12 f_0} \int_\Omega \left|\nabla u_{k^*}(x)\right|^2\, dx. \end{aligned} \tag{19}$$

On the other hand

$$H(W_{k^*}) \overset{(18)}{\leq} \frac{W_{k^*}}{4} \overset{(14)}{\leq} \frac{1}{8f_0} \int_\Omega |\triangle u_{k^*}(x)|^2\, dx. \tag{20}$$

Combining (19) and (20) we can conclude that

$$\begin{aligned} &H(W_n) + \sum_{k=k^*+1}^{n} \frac{h(W_{k-1})}{2\tau} \int_\Omega \left|\nabla(u_{k+1}(x) - u_k(x))\right|^2 dx \\ \leq\;& \frac{1}{8f_0} \int_\Omega \left(|\nabla u_{k^*}(x)|^2 + |\triangle u_{k^*}(x)|^2 \right) dx. \end{aligned} \tag{21}$$

Due to (12), we have also

$$\lim_{s\to\infty} H(s) \;=\; \frac{1}{2\delta_f^2} \overset{(11)}{=} \frac{\delta}{8f_0} \overset{(12)}{>} \frac{1}{8f_0} \int_\Omega \left(|\nabla u_{k^*}(x)|^2 + |\triangle u_{k^*}(x)|^2 \right) dx\,,$$

hence we obtain the following a priori estimate

$$W_k \leq C \qquad \text{for } k \geq k^*, \tag{22}$$

with C independent of k and n. At this point, (22) implies that

$$\int_\Omega \left| \frac{V_{k+1}(x) - V_k(x)}{\tau} \right|^2 dx \leq C \qquad \text{for } k \geq k^*. \tag{23}$$

On the other hand, using (14) and (21), we can conclude that

$$\frac{1}{\tau} \sum_{k=k^*}^{\infty} \int_\Omega |\nabla(u_{k+1}(x) - u_k(x))|^2\, dx \leq C.$$

Finally (23) and (6) give

$$\int_\Omega |\triangle u_k(x)|^2\, dx \leq C \qquad \text{for } k \geq k^*.$$

The rest of the proof follows as in Ref. [5], Section 4 (see Remark **2**.2). □

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A uniqueness result for a first order nonhomogeneous hyperbolic equation with hysteresis

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Abstract

We consider the Cauchy problem for a quasilinear hyperbolic equation with a possibly discontinuous hysteresis operator $\mathcal{F}$:

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] + \frac{\partial u}{\partial x} = f \qquad \text{in } R \times (0,T) \tag{1}$$

$$(u + w)(x, 0) = u^0 + w^0 \qquad \text{in } R. \tag{2}$$

For a general f uniqueness of the entropy solution is proved.
Keywords: Hysteresis, hyperbolic equation

1 Introduction

In this paper we consider the equation

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] + \frac{\partial u}{\partial x} = f \qquad \text{in } R \times (0,T) \tag{3}$$

$$(u + w)(x, 0) = u^0 + w^0 \qquad \text{in } R, \tag{4}$$

where $\mathcal{F}$ is a hysteresis operator, which we consider to be the discontinuous completed relay operator or its continuous regularization, see e.g. [3]. Existence of a weak solution of (3) was proved in [3] where also uniqueness was proved in the case when the source term vanishes. It was posed there as an open problem to prove the uniqueness without this (obviously restricting) assumption.

It is the aim of the present paper to show uniqueness of the weak solution constructed in [3] for a general non-zero right hand side f. We show that any limit of the time-discretized problems fulfils an entropy type condition (15). The classical argument of Kružkov based on doubling the variables then gives us Lipschitz continuous and monotone dependence of solutions on the initial data, hence uniqueness.

The equation (3) was also studied by A. Visintin in [5] where, by the method of nonlinear semigroup theory, he derived an existence and uniqueness of the so called integral solution. It was shown in [1] that this integral solution satisfies an entropy condition of the type introduced by Kružkov. Equation (3) was also studied by M. Peszyńska and R.E. Showalter in [2] with a hysteresis operator assumed to be a linear play operator.

We define $R_t := R \times (0,t)$ for any $t > 0$, fix any $T > 0$.

2 The main result

We consider the following weak formulation of the equation (3):

Problem 2.1. *Find $u_\epsilon \in L^\infty(0,T;L^2(R))$ and $w_\epsilon \in L^\infty(R_T)$ such that*

$$|w_\epsilon| \le 1 \ \text{a.e. in } R_T, \qquad \frac{\partial w_\epsilon}{\partial t} \in C^0(\overline{R_T})',$$

$$\begin{aligned} \iint_{R_T} \left((u_\epsilon + w_\epsilon - u^0 - w^0)\frac{\partial v}{\partial t} + u_\epsilon \frac{\partial v}{\partial x} + fv \right) dxdt &= 0 \\ \forall v \in H^1(R_T) \cap W^{1,1}(R_T), \quad v(\cdot,T) &= 0, \end{aligned} \tag{5}$$

$$\left.\begin{array}{l} (w_\epsilon - 1)(u_\epsilon - \epsilon w_\epsilon - \rho_2) \ge 0 \\ (w_\epsilon + 1)(u_\epsilon - \epsilon w_\epsilon - \rho_1) \ge 0 \end{array}\right\} \ \text{a.e. in } R_T,$$

$$\frac{1}{2}\int_R \left[u_\epsilon(x,t)^2 - u^0(x)^2\right]dx + \int_R \Psi_\rho^\epsilon\big(w_\epsilon,[0,t]\big)dx \le \iint_{R_T} f u_\epsilon dxdt \quad \text{for a.e. } t \in (0,T), \tag{6}$$

$$w_\epsilon(\cdot,0) = w^0 \qquad \text{a.e. in } R. \tag{7}$$

Let us set $s_0(\eta) := -1$ if $\eta < 0$, $s_0(0) := 0$, $s_0(\eta) := 1$ if $\eta > 0$, and

$$s_j(\zeta) := \max\{\min\{j\zeta, 1\}, -1\} \qquad \forall \zeta \in R, \ \forall j \in N. \tag{8}$$

Notice that $s_j \to s_0$ pointwise in R.

Theorem 2.2. *Let $\epsilon \ge 0$ and assume that*

$$u^0, w^0 \in L^2(R), \qquad |w^0| \le 1 \quad \text{a.e. in } R, \quad f \in L^1(R_T) \cap L^2(R_T). \tag{9}$$

Problem 1 has then a solution which satisfies for any $p \in [1,\infty)$:

$$\begin{gathered} u^0, w^0 \in BV(R), \qquad f \in L^p(0,T;BV(R)) \qquad \Rightarrow \\ u, w \in L^\infty(0,T;BV(R)) \cap W^{1,p}(0,T;C^0(\bar{R})'). \end{gathered} \tag{10}$$

Proof. (Sketch) Let us fix any $m \in N$, and set $h := \frac{T}{m}$,

$$u_m^0 := u^0, \quad w_m^0 := w^0, \quad f_m^n := \frac{1}{h}\int_{(n-1)h}^{nh} f(\cdot,t)dt \quad \text{for } n = 1,\dots,m.$$

We approximate our problem via an implicit time-discretization scheme.

Problem 2.3. *For $n = 1,\dots,m$, find $u_m^n \in H^1(R)$ and $w_m^n \in L^2(R)$ such that*

$$w_m^n \in G_\rho^\epsilon(u_m^n, w_m^{n-1}) \qquad a.e.\ in\ R,\ for\ n = 1,\dots,m,$$

$$\frac{u_m^n - u_m^{n-1}}{h} + \frac{w_m^n - w_m^{n-1}}{h} + \frac{du_m^n}{dx} = f_m^n \qquad a.e.\ in\ R,\ for\ n = 1,\dots,m. \quad (11)$$

Existence of an approximation solution can be proved step by step. For any family $\{v_m^n\}_{n=1,\dots,m}$ of functions $R \to R$, let us set

$$v_m := \text{ piecewise linear interpolate of } v_m^0,\dots v_m^n \text{ in } [0,T], \text{ a.e. in } R,$$
$$\bar{v}_m(\cdot,t) := v_m^n \quad \text{a.e. in } R, \forall t \in ((n-1)h, nh), \text{ for } n = 1,\dots,m.$$

After deriving appriori estimates, passage to the limit and using the discretized Hilpert inequality (see [6]) the following estimate can be derived:

$$\int_R (|\delta_k u_m^l| + |\delta_k w_m^l|)dx \le \int_R (|\delta_k u^0| + |\delta_k w^0|)dx + h\sum_{n=0}^{l}\int_R |\delta_k f_m^n|dx \quad (12)$$
$$\text{for } l = 1,\dots,m.$$

By the limit procedure we get the statement of the Theorem 1, for all details see [6]. □

Let us denote by $\mathcal{L}_\rho$ the hysteresis region.

Theorem 2.4. *Let $\epsilon \ge 0$ and assume that*

$$u^0, w^0 \in L^2(R), \qquad \exists\alpha \ge \tfrac{1}{2}: \qquad u^0, w^0 \in W^{\alpha,1}(R) \quad (13)$$
$$|w^0| \le 1 \quad a.e.\ in\ R, \quad f \in L^\infty(0,T;W^{\alpha,1}(R)). \quad (14)$$

Then there exists a solution of Problem 1 such that

$$\iint_{R_T} \Big((|u-\theta| + |w-\hat{\theta}|)\frac{\partial v}{\partial t} + |u-\theta|\frac{\partial v}{\partial x} + f s_0(u-\theta)v\Big)dxdt \ge 0 \quad (15)$$
$$\forall v \in \mathcal{D}(R_T), \qquad v \ge 0, \qquad \forall(\theta,\hat{\theta}) \in \mathcal{L}_\rho. \quad (16)$$

Proof. Let u_m^n, w_m^n, u and v be constructed via the approximation procedure in the proof of existence in Theorem 1. It can be shown in a similar way as in the existence proof of Visintin in [6] that

$$u_m \to u \qquad \text{strongly in } L^1_{\text{loc}}(R_T) \text{ and} \quad (17)$$
$$w_m \to w \qquad \text{strongly in } L^1_{\text{loc}}(R_T). \quad (18)$$

Let us assume further that $\theta \neq \rho_1, \rho_2$, so that there exists $\epsilon > 0$ such that k_ρ^ϵ maps θ to $\hat{\theta}$. Once we prove our statement for any pair $(\theta, \hat{\theta})$ like this, an obviuos approximation procedure will provide it for any $(\theta, \hat{\theta}) \in \mathcal{L}_\rho$.

Let us fix any nonnegative $v \in \mathcal{D}(R_T)$, set $v_m^n := \frac{1}{h}\int_{(n-1)h}^{nh} v(\cdot, t)dt$ a.e. in R and multiply the equation (11) with $hs_0(u_m^n - \theta)v_m^n$ and sum for $n = 1, \dots, m$. Since

$$(u_m^n - u_m^{n-1})s_0(u_m^n - \theta) \geq |u_m^n - \theta| - |u_m^{n-1} - \theta|$$

$$\frac{du_m^n}{dx} s_0(u_m^n - \theta) \geq \frac{d}{dx}|u_m^n - \theta|,$$

and by the discretized Hilpert inequality

$$(w_m^n - w_m^{n-1})s_0(u_m^n - \theta) \geq |w_m^n - \hat{\theta}| - |w_m^{n-1} - \hat{\theta}|, \tag{19}$$

after a continuous and discrete partial integration we obtain

$$h\sum_{n=0}^{m}\int_R \left(\left[|u_m^n - \theta| + |w_m^n - \hat{\theta}|\right] \frac{v_m^n - v_m^{n-1}}{h} + |u_m^n - \theta| \frac{dv_m^n}{dx} \right) dx \geq$$
$$\geq -h\sum_{n=0}^{m}\int_R f_m^n s_0(u_m^n - \theta)v_m^n dx \qquad \forall v \in \mathcal{D}(R_T), v \geq 0.$$

Passing to the limit as $m \to 0$ using that $u_m \to u$, $w_m \to w$ strongly in $L^1_{\text{loc}}(R_T)$ we get

$$\iint_{R_T} \left((|u - \theta| + |w - \hat{\theta}|)\frac{\partial v}{\partial t} + |u - \theta|\frac{\partial v}{\partial x} + f s_0(u - \theta)v \right) dxdt \geq 0,$$

where the limit in the last term is justified even if it contains a discontinuous function because the only case where a problem can arise is when $s_0(u - \theta) = 0$, t.m. $u = \theta$ and then since θ is a constant and u is a solution, f has to be equal to 0. This inequality holds for any $\epsilon > 0$. We now pass to the limit as $\epsilon \to 0$, as all estimates we derived are uniform w.r.t. ϵ, this finally yields (15) also for $\epsilon = 0$. □

Theorem 2.5. *(Lipschitz-Continuous and Monotone Dependence on Initial Data and right-hand side). Assume that* $\epsilon \geq 0$. *Let* u_i $i = 1, 2$ *be two solutions of Problem 1 with initial values* u_i^0 *satisfying the assumptions (13). Then*

$u_1 - u_2,\ w_1 - w_2 \in L^\infty(0,T;L^1(R))$, *and*

$$\int_R \big[|u_1(x,t)-u_2(x,t)|+|w_1(x,t)-w_2(x,t)|\big]dx \le \int_R \big[|u_1^0(x)-u_2^0(x)|+|w_1^0(x)-w_2^0(x)|\big]dx + \iint_{R_T} |f_1(x,t)-f_2(x,t)|dxdt \qquad \text{for a.a. } t\in(0,T),$$

$$\int_R \big[(u_1(x,t)-u_2(x,t))^+ +(w_1(x,t)-w_2(x,t))^+\big]dx \le \int_R \big[(u_1^0(x)-u_2^0(x))^+ +(w_1^0(x)-w_2^0(x))^+\big]dx + \iint_{R_T} (f_1(x,t)-f_2(x,t))^+ dxdt \qquad \text{for a.a. } t\in(0,T).$$

Proof. The proof is based on Kružkov's technique of doubling variables. We write the inequality (15) for $(u_1(x,t), w_1(x,t))$ and $(\theta,\bar\theta)=(u_2(\xi,\tau),w_2(\xi,\tau))$ for almost any fixed $(\xi,\tau)\in R_T$ and $f=f_1$ and the same inequality for $(u_2(\xi,\tau),w_2(\xi,\tau))$ and $(\theta,\bar\theta)=(u_1(x,t),w_1(x,t))$ for almost any fixed $(x,t)\in R_T$ and $f=f_2$, take any nonnegative $v=v(x,t,\xi,\tau)\in\mathcal{D}((R_T)^2)$ in both of the above inequalities and then integrate the first one w.r.t. (ξ,τ) and the second one w.r.t. (x,t) over R_T. After summing these two inequalities we get:

$$\iiiint_{(R_T)^2}\Big[\big(|u_1(x,t)-u_2(\xi,\tau)|+|w_1(x,t)-w_2(\xi,\tau)|\big)\Big(\frac{\partial v}{\partial t}+\frac{\partial v}{\partial \tau}\Big) + |u_1(x,t)-u_2(\xi,\tau)|\Big(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial \xi}\Big) + (f_1-f_2)s_0(u_1(x,t)-u_2(\xi,\tau))v\Big]dxdtd\xi d\tau \ge 0.$$

The rest of the proof is a standard application of Kružkov's technique and can be done in the same way as in [3]. □

We close the paper with an existence and uniqueness result for the following nonlinear equation

$$\frac{\partial}{\partial t}[u+\mathcal{F}(u)]+\frac{\partial u}{\partial x}=f(x,t,u) \qquad \text{in } R\times(0,T) \tag{20a}$$

$$(u+w)(x,0)=u^0+w^0 \qquad \text{in } R. \tag{20b}$$

The result will follow from Theorem 2.5 and a Fixed-point theorem.

Theorem 2.6. *Suppose that* $f: R\times(0,T)\times L^1(R_T)\to L^1(R_T)$ *be a Lipschitz continuous function with a Lipschitz constant L in u in the following sense*

$$\iint_{R_T}|f(x,t,u_1)-f(x,t,u_2)|dxdt \le \iint_{R_T}|u_1-u_2|dxdt.$$

Then there exists a unique solution u of (20) such that $u\in L^1(R_T)$.

Proof. For any $v \in L^1(R_T)$, let u be the solution of the equation

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] + \frac{\partial u}{\partial x} = f(x,t,v) \qquad \text{in } R \times (0,T). \tag{21}$$

which satisfies the given initial condition. This defines an operator $J : L^1(R_T) \to L^1(R_T) : v \to u$. For $v_1, v_2 \in L^1(R_T)$ it follows from the Lipschitz continuity of the function f in u and Theorem 2.5 that

$$\begin{aligned}
&\int_R |u_1(x,t) - u_2(x,t)|dx \\
&\qquad \le -\iint_{R_T} [f_1(x,t,v_1) - f_2(x,t,v_2)][s_0(u_1(x,t) - u_2(x,t))]dxdt \\
&\qquad \le L \iint_{R_T} |v_1(x,t) - v_2(x,t)|.
\end{aligned}$$

This gives after integration in $t \in [0,T]$

$$\iint_{R_T} |u_1(x,t) - u_2(x,t)|dx \le LT \iint_{R_T} |v_1(x,t) - v_2(x,t)|.$$

Now, for $TL \le 1$, J is a contraction, hence it has one and only one fixed point, which is the solution of (20). If instead $TL \ge 1$, we then divide the interval $[0,T]$ into a finite number of subintervals of lenght smaller than L^{-1} and apply the previous procedure step by step. This yields existence and uniqueness of solution in $[0,T]$. □

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On an evolutionary process with linearly growing energy

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Abstract

We formulate a rate-independent evolution problem inspired by problems in the deformation theory of plasticity. The stored energy density depends on the gradient; it does not have to be quasiconvex and is assumed to exhibit linear growth at infinity.
Keywords: Concentrations, oscillations, rate-independent evolution.

1 Introduction

In this note, we formulate a rate-independent mesoscopic process where the evolution is not triggered by applied forces as in [3] but by time-dependent Dirichlet boundary conditions. We formulate the problem here, a detailed mathematical analysis regarding the existence of a solution can be found in [2].

We sketch the problem that motivates our investigation; details can be found elsewhere [3]. Crystalline materials can often be characterised via energy minimisation; for plastically deformed crystals, Ortiz and Repetto [6] provide a setting in which dislocation structures can be described by a nonconvex minimisation problem. The nature of this variational model is incremental, to reflect the irreversible nature of plastic deformations [6]. We account for these phenomena with a phenomenological dissipation functional. We consider an energy that depends on a strain tensor. A characteristic feature of the problem is that the stored energy has linear growth at infinity; this is motivated in greater detail in [3].

In the following discussion, $\Omega \subset \mathbb{R}^n$ is always a bounded domain with smooth boundary, $M(\cdot)$ denotes the space of regular Borel measures and $W^{1,1}_{u_D}(\Omega;\mathbb{R}^m)$ stands for the set of functions $u \in W^{1,1}(\Omega;\mathbb{R}^m)$ with $u = u_D$ on $\partial\Omega$. Here $u_D \in W^{1,1}(\Omega;\mathbb{R}^m)$ is given. It is well known that $W^{1,1}(\Omega;\mathbb{R}^m)$ is non-reflexive, that is, a bounded sequence does not necessarily contain a subsequence with a weak limit in $W^{1,1}(\Omega;\mathbb{R}^m)$. Hence, one looks for an extension of $W^{1,1}(\Omega;\mathbb{R}^m)$. Instead of the usual space of functions of bounded variations, we will work with the so-called Souček space [7]; we denote it by $W^{1,\mu}(\bar\Omega;\mathbb{R}^m)$. This extension consists of functions in $L^1(\Omega;\mathbb{R}^m)$ whose gradient is a measure on $\bar\Omega$. The precise formulation is as follows.

$$\begin{aligned} W^{1,\mu}(\Omega;\mathbb{R}^m) = \Big\{ &(u,\bar D u) \in L^1(\Omega;\mathbb{R}^m) \times M(\bar\Omega;\mathbb{R}^{m\times n}); \\ &\text{there exists } \{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}(\Omega;\mathbb{R}^m) \text{ such that} \\ &u_k \to u \text{ in } L^1(\Omega;\mathbb{R}^m) \text{ and } \nabla u_k \to \bar D u \text{ weakly* in } M(\bar\Omega;\mathbb{R}^{m\times n})\Big\}. \end{aligned}$$

It is known [7] that $W^{1,\mu}(\Omega;\mathbb{R}^m)$ is a Banach space if equipped with the norm

$$\left\|(u,\bar D u)\right\|_{W^{1,\mu}(\Omega;\mathbb{R}^m)} = \|u\|_{L^1(\Omega;\mathbb{R}^m)} + \left\|\bar D u\right\|_{M(\bar\Omega;\mathbb{R}^{m\times n})}.$$

The weak* convergence in $W^{1,\mu}(\Omega;\mathbb{R}^m)$ is defined analogously to $BV(\Omega;\mathbb{R}^m)$; the precise formulation can be found in the literature [7, 3]. Moreover, as shown in [7, Theorem 1 (iii)], if $(u, Du) \in W^{1,\mu}(\Omega;\mathbb{R}^m)$, then there is a unique measure $\bar T(u,\bar D u) \in M(\partial\Omega;\mathbb{R}^m)$ such that

$$\int_{\partial\Omega} (\varphi\cdot\nu)\left[\bar T(u^j,\bar D u^j)\right](\mathrm{d}A) = \int_\Omega u^j(x)\,\mathrm{div}\,\varphi(x)\,\mathrm{d}x + \int_{\bar\Omega} \varphi\cdot\left[\bar D u^j\right](\mathrm{d}x) \qquad (1)$$

for all $\varphi \in C^1(\bar\Omega;\mathbb{R}^n)$ and all $1 \le j \le m$. The measure

$$\bar T(u,\bar D u) = \left(\bar T(u^1,\bar D u^1),\ldots,\bar T(u^m,\bar D u^m)\right)$$

is called the *trace* of $(u,\bar D u$. Here, the measure $\bar D u^j$ denotes the jth row of the matrix-valued measure $\bar D u$. The operator $W^{1,\mu}(\Omega;\mathbb{R}^m) \to M(\partial\Omega;\mathbb{R}^m)$ given by $(u, Du) \mapsto \bar T u$ is (weak*, weak*) continuous [7, Theorem 2 (ii)]. Finally, balls in $W^{1,\mu}(\Omega;\mathbb{R}^m)$ are weakly* compact [7, Theorem 6]. The following Poincaré-type inequality has been proved recently [3].

Lemma 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with $\partial\Omega$ belonging to class C^1. Let $\Gamma_D \subset \partial\Omega$ be relatively open and of positive $(n-1)$- dimensional Lebesgue measure; suppose further that $z \in M(\Gamma_D;\mathbb{R}^m)$. Then there is $C > 0$ such that the estimate*

$$\|u\|_{L^1(\Omega;\mathbb{R}^m)} \le C\left(\left\|\bar D u\right\|_{M(\bar\Omega;\mathbb{R}^{m\times n})} + \|z\|_{M(\Gamma_D;\mathbb{R}^m)}\right) \qquad (2)$$

holds for all $(u,\bar D u) \in W^{1,\mu}(\Omega;\mathbb{R}^m)$ with $\bar T(u,\bar D u) = z$ on Γ_D.

We will also need a generalisation of Young measures, due to DiPerna and Majda [1]. Let us consider a completely regular subalgebra $\mathcal{F}$ of the space of bounded continuous functions $BC(\mathbb{R}^d)$ (details are presented elsewhere [3]); in the application, we will set $d := mn$. We require that $\mathcal{F}$ contains all functions $\tilde{f}$ for which the radial limit $\lim_{r\to\infty} \tilde{f}(rs)$ exists for arbitrary $s \in \mathbb{R}^d$. We note that $\mathcal{F}$ also may contain functions $\tilde{f}$ which have no well-defined radial limits. To deal with functions f with linear growth at infinity in a convenient manner, we set $\tilde{f}(s) := \frac{f(s)}{1+|s|}$, with $\tilde{f} \in \mathcal{F}$.

For a bounded sequence $\{u_k\}_{k\in\mathbb{N}}$ in $L^1(\bar{\Omega};\mathbb{R}^d)$, there exists a non-negative Radon measure $\sigma \in M^+(\bar{\Omega})$ such that [1, Theorem 4.1]

$$(1 + |u_k(x)|)\,\mathrm{d}x \overset{\star}{\rightharpoonup} \sigma \text{ in } M(\bar{\Omega}). \tag{3}$$

Furthermore, for a separable completely regular subalgebra $\mathcal{F}$ of $BC(\mathbb{R}^d)$, a σ-measurable map $\hat{\nu}\colon \Omega \to \mathrm{Prob}(\beta_{\mathcal{F}}\mathbb{R}^d)$, $x \mapsto \hat{\nu}_x$ exists, and a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ (not relabelled) such that for every $\tilde{f} \in \mathcal{F}$

$$\lim_{k\to\infty} \int_{\bar{\Omega}} \phi(x) f(u_k(x))\,\mathrm{d}x = \int_{\bar{\Omega}} \phi(x) \int_{\beta_{\mathcal{F}}\mathbb{R}^d} \tilde{f}(s)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x) \tag{4}$$

holds for every $\phi \in C(\bar{\Omega})$ [1, Theorem 4.3]. We say that $\{u_k\}_{\in\mathbb{N}}$ *generates* the pair $(\sigma, \hat{\nu})$ if Equation (4) holds. A pair

$$(\sigma, \hat{\nu}) \in M^+(\bar{\Omega}) \times L^\infty_{\mathrm{w}}\left(\bar{\Omega}, \sigma; \mathrm{Prob}(\beta_{\mathcal{F}}\mathbb{R}^d)\right)$$

attainable by sequences in $L^1(\Omega;\mathbb{R}^d)$ is called a *DiPerna-Majda measure*. The set of all DiPerna-Majda measures is denoted $\mathcal{DM}_{\mathcal{F}}(\Omega;\mathbb{R}^d)$. We denote by $\mathcal{GDM}_{\mathcal{F}}(\Omega;\mathbb{R}^{m\times n})$ the subset of $\mathcal{DM}_{\mathcal{F}}(\Omega;\mathbb{R}^{m\times n})$ of those measures which are generated by gradients of mappings in $W^{1,1}(\Omega;\mathbb{R}^m)$.

2 Static problem

We first discuss a static variational problem with a linear growth energy. The energy is assumed to be a continuous function $W\colon \bar{\Omega} \times \mathbb{R}^{n\times m} \to \mathbb{R}$ such that constants $\beta \geq \alpha > 0$ exist with

$$\alpha\,(|s| - 1) \leq W(x, s) \leq \beta\,(1 + |s|) \text{ for every } x \in \bar{\Omega}. \tag{5}$$

The variational problem is then to

$$\text{minimise } I(u) := \int_{\Omega} W(x, \nabla u(x))\,\mathrm{d}x \text{ among } u \in W^{1,1}_{u_D}(\Omega;\mathbb{R}^m). \tag{6}$$

In general, there is no solution to (6), because of the non-reflexivity of the underlying space and the possible non-(quasi)convexity of $W(x,\cdot)$. In order to capture the limiting behaviour of minimising sequences, we state the following relaxed problem:

$$\text{Minimise}\quad \bar{I}\left(u,\bar{D}u,\sigma,\hat{\nu}\right) := \int_{\bar{\Omega}}\int_{\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \tilde{W}(x,s)\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x) \tag{7}$$

$$\text{among}\quad \left(u,\bar{D}u\right)\in W^{1,\mu}(\Omega;\mathbb{R}^m)\ \text{with}\ \bar{T}\left(u,\bar{D}u\right)=u_D\ \text{on}\ \partial\Omega,$$

$$\text{and}\quad (\sigma,\hat{\nu})\in \mathcal{GDM}^{u_D}_{\mathcal{F}}(\Omega;\mathbb{R}^{m\times n}),\ \text{where}\ \bar{D}u\ \text{is given by}$$

$$\int_{\bar{\Omega}}\phi(x)\bar{D}u\,\mathrm{d}x = \int_{\bar{\Omega}}\phi(x)\int_{\beta_{\mathcal{F}}\mathbb{R}^{m\times n}}\frac{s}{1+|s|}\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x). \tag{8}$$

It can be shown [3] that (7) has a solution and $\min \bar{I} = \inf I$. Moreover, minimising sequences of I generate (in the sense of (4)) minimisers of $\bar{I}$ and every minimiser of $\bar{I}$ is generated by a minimising sequence of I.

3 Evolution

We now turn our attention to the analysis of the evolution during an arbitrary, but fixed time interval $[0,T]$. The evolution will be triggered by changes in the Dirichlet boundary data. To account for the energy that may be dissipated during the evolution, we follow Mielke and co-workers [5, 4] in introducing a *dissipation distance*. As for the force-driven evolution [3], we define the (mesoscopic) dissipation distance between two DiPerna-Majda measures $\eta_1,\eta_2\in GDM^{u_D}_{\mathcal{F}}(\Omega;\mathbb{R}^{m\times n})$, since these measures record the microstructure. We formulate the evolutionary problem here and remark that an existence theory requires further assumptions on the regularity and growth of the time-dependent energy; these assumptions can be found in [2].

Definition 3.1. *The dissipation* D *has to satisfy the following conditions.*

1. *The triangle inequality is valid for* D*. That is, for any three internal states* η_1,η_2,η_3*, it holds that*

$$\mathrm{D}\left(\eta_1,\eta_3\right)\le \mathrm{D}\left(\eta_1,\eta_2\right)+\mathrm{D}\left(\eta_2,\eta_3\right). \tag{9}$$

2. *We suppose that there is* $L\in\mathbb{N}$ *and a continuous bounded mapping* $\Lambda:\mathbb{R}^{m\times n}\to\mathbb{R}^L$ *such that* $\Lambda_j\in\mathcal{F}$ *for* $1\le j\le L$ *such that the mesoscopic order parameter* λ *associated with the system configuration described by* $(u,\bar{D}u,\sigma,\hat{\nu})$ *is given by the formula*

$$\lambda := \int_{\beta_{\mathcal{F}}\mathbb{R}^{m\times n}}\Lambda(s)\hat{\nu}_x(\mathrm{d}s)\sigma, \tag{10}$$

which means that $\lambda\in M(\bar{\Omega};\mathbb{R}^L)$ *is a measure such that, for all* $g\in C(\bar{\Omega})$,

$$\int_{\bar{\Omega}} g(x)\lambda(\mathrm{d}x) = \int_{\bar{\Omega}}\int_{\beta_{\mathcal{F}}\mathbb{R}^{m\times n}}\Lambda(s)\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x).$$

Specifically, we write

$$\mathrm{D}\left(\eta_1,\eta_2\right) = \left\|\lambda_1-\lambda_2\right\|_{M(\bar{\Omega};\mathbb{R}^L)}\ . \tag{11}$$

Now we are in a position to define the set of admissible configurations. Each such configuration will be written as $q := (u, \bar{D}u, \eta, \lambda)$. The set of admissible configurations is then [2]

$$Q := (u_D, \bar{D}u_D, \eta_D, \lambda_D) + Q_0 \ , \tag{12}$$

with Q_0 being is the set of admissible configurations with homogeneous Dirichlet data,

$$\begin{aligned} Q_0 := \Big\{ q_0 = \big(u_0, \bar{D}u_0, \eta_0, \lambda_0\big) \ \text{with} \\ \big(u_0, \bar{D}u_0\big) \in W^{1,\mu}(\Omega; \mathbb{R}^m), \eta_0 \in GDM_{\mathcal{F}}^{u_D}(\Omega; \mathbb{R}^{m\times n}), \lambda_0 \in M\left(\bar{\Omega}; \mathbb{R}^L\right), \\ \bar{D}u_0 = \mathrm{Id} \bullet \eta_0, \ \lambda \text{ given by (10), and } \bar{T}\left(u_0, \bar{D}u_0\right) = 0 \text{ on } \Gamma_D \Big\}. \end{aligned}$$

Though Q depends on time, this is suppressed from the notation.

For convenience, we write $\mathrm{D}(q_1, q_2) := \mathrm{D}(\eta_1, \eta_2)$. Further, let us abbreviate $\Gamma(t, q) := \int_{\bar{\Omega}\times\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \tilde{W}(x, s)$. Finally, for a process $q\colon [0, T] \to Q$ and a given time interval $[t_1, t_2] \subset [0, T]$, the *temporal dissipation* is given by

$$\mathrm{Diss}\,(q, [t_1, t_2]) := \sup_{J\in\mathbb{N}} \left\{ \sum_{j=1}^{J} \mathrm{D}\left(\eta(\tau_{j-1}), \eta(\tau_j)\right) \ \Big| \ t_1 = \tau_0 < \dots < \tau_J = t_2 \right\}.$$

Definition 3.2. *Given $q_0 \in Q$, we say that the process $q\colon [0, T] \to Q$ is a* solution *if the following conditions hold:*

1. $\big(u, \bar{D}u\big) \in L^\infty(0, T; W^{1,\mu}(\Omega; \mathbb{R}^m))$,
2. $\lambda \in BV\left(0, T; L^1(\Omega; \mathbb{R}^L)\right)$.
3. Global Stability: *For every $t \in [0, T]$, the process is stable in the global sense,*
$$\Gamma(t, q(t)) \le \Gamma(t, \tilde{q}) + \mathrm{D}\left(q(t), \tilde{q}\right) \ \textit{for every}\ \tilde{q} \in Q. \tag{13}$$
4. Energy inequality: *For every $0 \le t_1 \le t_2 \le T$, we have*
$$\Gamma\left(t_1, q(t_1)\right) + \mathrm{Diss}\left(q, [t_1, t_2]\right) \le \Gamma\left(t_2, q(t_2)\right) - \int_{t_1}^{t_2} \int_{\partial\Omega} \partial_t \Gamma(r, q(r))\, \mathrm{d}S\, \mathrm{d}t,$$
5. Initial condition: $q(0) = q_0$ *and* $\Gamma(0, q(0)) < \infty$.

An energetic solution for a suitable $u_D \in C^1([0, T]; W^{1,1}(\Omega; \mathbb{R}^m))$ can be approximated in a constructive way, using a sequence of incremental problems. For a given initial condition $q_\tau^0 = q_0$, it is natural to define q_τ^k for $k = 1, \dots, N$ as a solution to the problem

$$\min_{q\in Q} \Gamma(k\tau, q) + \mathrm{D}(q_\tau^{k-1}, q). \tag{14}$$

A precise existence statement and its proof will be published in [2].

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On an evolutionary model for complete damage based on energies and stresses

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Abstract

A recent model [BMR09] allows for complete damage, such that the deformation is not well-defined. The evolution can be described in terms of energy densities and stresses. We introduce the notion of *generalized energetic solution* and show how the existence theory can be generalized to convex, but non-quadratic elastic energies. We also discuss Γ-convergence from partial to complete damage.

Keywords: generalized energetic solution, rate independent energetic system, complete damage, Gamma convergence.

1 Introduction

There is a rich literature [Ort85, FrM93, DPO94, FrN96, DMT01, MaA01, HaS03] on rate-independent mechanical models for damage in brittle materials, and recently several mathematical approaches [FrM98, FKS99, FrG06] were developed, in particular the abstract theory of rate-independent processes [MiT99, MiT04, Mie05] proved very helpful as it allows one to employ the machinery of incremental minimization.

Here we want to contribute to the models discussed in [MiR06, BMR09, MRZ10]. Let $u : \Omega \to \mathbb{R}^d$ be the displacement and $z : \Omega \to [0, 1]$ the damage variable, then the rate-independent system is given by the triple $(\mathcal{F}\times\mathcal{Z}, \mathcal{E}, \mathcal{D})$, where $u \in \mathcal{F}$, $z \in \mathcal{Z}$. The energy-storage functional has the form

$$\mathcal{E}_\delta(t, u, z) = \int_\Omega W(x, \mathbf{e}(u_\mathrm{D}(t)+u)(x), z(x)) + \delta|\mathbf{e}(u_\mathrm{D}(t)+u)|^p \,\mathrm{d}x + \mathcal{G}(z),$$

and the dissipation is $\mathcal{D}(z, \widehat{z}) = \int_\Omega D(x, z(x), \widetilde{z}(x))\,\mathrm{d}x$. For $\delta > 0$ existence of energetic solutions (u_δ, z_δ) is known [MiR06] for general W. The limit passage for $\delta \to 0$ in the sense of Γ-limits works under the assumption that $e \mapsto W(x, e, z)$ is quadratic [BMR09, MRZ10]. The difficulty is that W is not coercive, hence in the limit $\delta \to 0$ we are not able to control u_δ, and convergence should only be valid for z_δ. The task is to define a limit equation in terms of z. In particular one needs a replacement of the power of the external forces giving the limit of $\partial_t \mathcal{E}_\delta(u_\delta(t), z_\delta(t))$.

Here we discuss the changes needed to generalize from a quadratic $W(x,\cdot,z)$ to arbitrary strictly convex potentials with p growth from above. The main idea is to use a reduced functional $\mathcal{I}_\delta(t,z)$ avoiding the usage of u; however, to keep control over stresses one introduces an auxiliary functional $\mathcal{V}_\delta : \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\times\mathcal{Z} \to \mathbb{R}$ such that

$$\mathcal{I}_\delta(t,z) = \min\{\, \mathcal{E}_\delta(t,\widetilde{u},z) \mid u \in \mathcal{F}\,\} = \mathcal{V}_\delta(\mathbf{e}(u_{\mathrm{D}}(t)), z) + \mathcal{G}(z),$$

and $\mathrm{D}_e\mathcal{V}_\delta(\mathbf{e}(u_{\mathrm{D}}(t)), z(t)) \in \mathrm{L}^{p/(p-1)}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ gives the equilibrium stress.

In $(\mathcal{Z},\mathcal{I}_\delta,\mathcal{D})$ it is possible to pass to the Γ-limit for $\delta \to 0$ with respect to the weak convergence in $\mathcal{Z} \subset \mathrm{W}^{1,r}(\Omega)$. However, the Γ-limit $\mathfrak{I}(t,\cdot)$ loses in general differentiability in t, since we are not able to show that the Γ-limit $\mathfrak{V}(e,\cdot)$ of $\mathcal{V}_\delta(e,\cdot)$ remains differentiable in e. Nevertheless, the convexity of $\mathfrak{V}(\cdot,z)$ allows us to characterize the Clarke differential using the left and right partial derivative in t:

$$\partial_t^{\mathrm{Cl}}\mathfrak{I}(t,z) = \Big[\partial_t^-\mathfrak{I}(t,z), \partial_t^+\mathfrak{I}(t,z)\Big],$$

where $\partial_t^\pm\mathfrak{I}(t,z) = \pm\sup\big\{\, \langle \pm\sigma, \mathbf{e}(\dot{u}_{\mathrm{D}}(t))\rangle \mid \sigma \in \partial_e^{\mathrm{sub}}\mathfrak{V}(\mathbf{e}(u_{\mathrm{D}}(t)), z)\,\big\}$.

We generalize the notion of energetic solutions [Mie05] to *generalized energetic solutions* by keeping stability (S) and replacing the energy balance by

$$\mathfrak{I}(t,z(t)) + \mathrm{Diss}_{\mathcal{D}}(z,[0,t]) = \mathfrak{I}(0,z(0)) + \int_0^t p(\tau)\,\mathrm{d}\tau,$$

where p has to satisfy $p(\tau) \in \partial_\tau^{\mathrm{Cl}}\mathfrak{I}(\tau,z(\tau))$ a.e. in $[0,T]$, see Definition **4**.3. Theorem **4**.4 establishes existence of such generalized energetic solutions to $(\mathcal{Z};\mathfrak{I},\mathcal{D})$. Moreover, assuming that a certain conjecture holds, we show that a subsequence $(z_{\delta_j})_{j\in\mathbb{N}}$ converges to weak energetic solution for $(\mathcal{Z},\mathfrak{I},\mathcal{D})$.

2 Setup of the model

The body $\Omega \subset \mathbb{R}^d$ is described by a bounded Lipschitz domain. The state of the system is described by the displacement $\widetilde{u} : \Omega \to \mathbb{R}^d$ and the scalar damage variable $z : \Omega \to [0,1]$, where $z = 1$ denotes no damage and $z = 0$ means that the maximal damage has been reached (all microscopic breakable structures are broken). The displacement $\widetilde{u}$ will satisfy time-dependent Dirichlet boundary conditions on $\Gamma_{\mathrm{D}} \subset \partial\Omega$ via $u_{\mathrm{D}} \in \mathrm{C}^1([0,T],\mathrm{W}^{1,p}(\Omega))$ in the form

$$\widetilde{u}(t) = u_{\mathrm{D}}(t) + u(t) \quad \text{with } u(t) \in \mathcal{F} = \big\{\, v \in \mathrm{W}^{1,p}(\Omega) \mid v|_{\Gamma_{\mathrm{D}}} \equiv 0\,\big\}\,.$$

We also use the infinitesimal strain tensor $\mathbf{e}(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^{\mathsf{T}}\big)$ and set

$$\mathbf{e}_{\mathrm{D}}(t) = \mathbf{e}(u_{\mathrm{D}}(t)) \text{ and } \dot{\mathbf{e}}_{\mathrm{D}}(t) = \mathbf{e}(\dot{u}_{\mathrm{D}}(t)) \text{ where } \dot{} = \partial_t.$$

The stored energy of the system is given via the functional

$$\begin{aligned}&\mathcal{E}(t,u,z) = \int_\Omega W(x, \mathbf{e}_\mathrm{D}(t,x){+}\mathbf{e}(u)(x), z(x))\,\mathrm{d}x + \mathcal{G}(z)\\ &\text{with } \mathcal{G}(z) = \int_\Omega b(z(x)) + \kappa(x)|\nabla z(x)|^r\,\mathrm{d}x,\end{aligned}$$

where $\kappa \in \mathrm{L}^\infty(\Omega)$ and $\kappa(x) \geq c_0$ a.e. Thus, the suitable space for the deformation states is $\mathcal{Z} = \big\{\, z \in \mathrm{W}^{1,r}(\Omega) \mid 0 \leq z \leq 1 \,\big\}$. The additional term b is intended to model cohesive effects and should satisfy $b'(z) \leq 0$, i.e., if the stresses in the material are released then the damage may heal ($\dot z > 0$) by using up some energy.

The stored energy density $W : \Omega \times \mathbf{E}_d \times [0,1] \to \mathbb{R}$, where $\mathbf{E}_d = \mathbb{R}^{d\times d}_\mathrm{sym}$, is a Carathéordory function satisfying

$$\forall\,(x,z) \in \Omega: \quad W(x,\cdot,z) \in \mathrm{C}^1(\mathbf{E}_d), \tag{1a}$$
$$\exists\, C>0\ \forall\,(x,e,z): \quad W(x,e,z) \leq C|e|^p + C, \tag{1b}$$
$$\forall\,(x,z): \quad e \mapsto W(x,e,z) \text{ is strictly convex}, \tag{1c}$$
$$\forall\,(x,e): \quad z \mapsto W(x,e,z) \text{ is nondecreasing}, \tag{1d}$$
$$\exists\, c_1,c_2\ \forall\,(x,e,z): \ |\partial_e W(x,e,z)| \leq c_1(W(x,e,z){+}c_2)^{1-1/p}. \tag{1e}$$

Condition (1d) means that the material becomes weaker if damage increases, and (1e) is called “stress control”, since it allows us to control the size of the stresses in terms of the energy alone, uniformly in (x,z). A typical function W has the form

$$W(x,e,z) = W_0(x,e) + a(z)W_1(x,e),$$

where W_0 and W_1 are smooth and convex, W_0 may be non-coercive while W_1 is coercive, $a(z) \geq cz^\alpha$ and $a'(z) \geq 0$.

Finally we describe the dissipation functional $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ via

$$\mathcal{D}(z_0,z_1) = \int_\Omega D(x, z_0(x), z_1(x))\,\mathrm{d}x,$$

where, for each x, D satisfies the triangle inequality and the coercivity $D(x,z,\widetilde z) \geq C|z{-}\widetilde z|$. The typical choice is $D(x,z,\widetilde z) = \delta_+(z{-}\widetilde z)$ for $\widetilde z \leq z$ and $\delta_-(\widetilde z{-}z)$ for $z \leq \widetilde z$, where $\delta_+ \in (0,\infty)$ and $\delta_- \in (0,\infty]$. Here $\delta_- = \infty$ forbids healing, which can only take place if $\delta_- + b'(z) < 0$ for some z.

With these functionals we define notion of energetic solution [MiT99, MiT04] (see also the surveys [Mie05, MiR09]) for the rate-independent energetic system $(\mathcal{Q},\mathcal{E},\mathcal{D})$, where $\mathcal{Q} = \mathcal{F}\times\mathcal{Z}$. A mapping $q = (u,z) : [0,T] \to \mathcal{Q}$ is called **energetic solution** if for all $t \in [0,T]$ we have **stability (S)** and **energy balance (E)**:

$$\begin{aligned}&\textbf{(S)} \quad \forall\,\widetilde q = (\widetilde u, \widetilde z) \in \mathcal{Q}: \quad \mathcal{E}(t,q(t)) \leq \mathcal{E}(t,\widetilde q) + \mathcal{D}(z(t),\widetilde z);\\ &\textbf{(E)} \quad \mathcal{E}(t,q(t)) + \mathrm{Diss}_\mathcal{D}(z,[0,t]) = \mathcal{E}(0,q(0)) + \int_0^t \partial_\tau \mathcal{E}(\tau,q(\tau))\,\mathrm{d}\tau.\end{aligned} \tag{2}$$

Here $\mathrm{Diss}_\mathcal{D}(z,[r,s])$ is defined to be the supremum of $\sum_1^N \mathcal{D}(z(t_{j-1},z(t_j))$ over all finite partitions $r \leq t_0 < t_1 \cdots t_N \leq s$. Here we use that for each q the power of the external forces $\partial_t \mathcal{E}(t,q)$ is well defined by using (1e), and (E) implicitly assumes that $t \mapsto \partial_t \mathcal{E}(t,q(t))$ is measurable.

For non-coercive problems, where u is no longer well-defined, we will see that it is the main problem how to define this partial derivative $\partial_t \mathcal{E}(t,q)$. Thus, it is an open problem whether under the above assumption a general existence result holds. However, the coercive case was solved under more general assumptions including unilateral constraints and volume forces [MiR06]. To make this theory applicable we strengthen the lower bound in (1b) to make it coercive in e for all $z \in [0,1]$.

Theorem 2.1. *If the above assumption hold with $p > 1$ and $r > d$ and if W additionally satisfies*

$$\exists\, C, c > 0\; \forall\, (x,e,z): \quad c|e|^p - C \le W(x,e,z),$$

then for all stable initial states $q_0 \in \mathcal{Q}$ (i.e., (S) holds at $t = 0$ with $q(0)$ replaced by q_0) there exists an energetic solution $q : [0,T] \to \mathcal{Q}$ with $q(0) = q_0$ and $q \in \mathrm{L}^\infty([0,T], \mathrm{W}^{1,p}(\Omega) \times \mathrm{W}^{1,r}(\Omega)$ and $z \in \mathrm{B}([0,T], \mathrm{W}^{1,r}(\Omega))$.

3 Reformulation based on stress and energy

The approach for solving non-coercive problems was indicated already in [MiR06] and finally solved in [BMR09] under the additional assumption that W is quadratic: $W(x,e,z) = \frac{z}{2} e{:}\mathbb{C}{:}e$; however more general quadratic forms $\frac{1}{2} e{:}\mathbb{C}(z){:}e + g(z){:}e + \gamma(z)$ would work equally well. The main idea is to approximate the non-coercive case with a coercive one by setting

$$W_\delta(x,e,z) = W(x,e,z) + \delta(1+|e|^2)^{p/2}. \tag{3}$$

Then for each $\delta > 0$ there is a solution $q_\delta = (u_\delta, z_\delta)$ of the rate-independent energetic system $(\mathcal{Q}, \mathcal{E}_\delta, \mathcal{D})$. Moreover, using the stress control (1e) it is not difficult to show that there exists $C > 0$ such that for all $\delta \in (0,1)$ and all $t \in [0,T]$ we have $\mathcal{E}_\delta(t, q_\delta(t)) + \mathrm{Diss}_{\mathcal{D}}(z_\delta, [0,t]) \le C$.

Now, using the theory of Γ-convergence of rate-independent energetic systems [MRS08] it is then possible to pass to the limit in the reduced system, where the displacement u is minimized out. The latter step is essential, since it is not to be expected that u_δ or $\mathbf{e}(u_\delta)$ converges in any reasonably sense. In regions where $z = 0$ holds we may have $W(x,e,0) = 0$ for a large and possibly unbounded set of strains $e \in \mathbf{E}_d$ due to the missing coercivity.

To define the reduced problem we use the strict convexity (1c) to find that $\mathcal{E}_\delta(t,\cdot,z)$ has a unique minimizer $u = U_\delta(t,z) \in \mathcal{F}$. With this we have

$$\mathcal{I}_\delta(t,z) = \int_\Omega W_\delta(x, \mathbf{e}_{\mathrm{D}}(t) + \mathbf{e}(U_\delta(t,z)), z)\,\mathrm{d}x + \mathcal{G}(z).$$

A classical argument [KnM08, KMZ08] shows that $\partial_t \mathcal{I}_\delta(t,z) = \partial_t \mathcal{E}_\delta(t, U_\delta(t,z), z)$.

While the limit of the energy $\mathcal{I}_\delta(t,z_\delta)$ along energetic solutions q_δ can be understood in the sense of Γ-limits, it is nontrivial to control the power

$$\begin{aligned}&\partial_t\mathcal{I}_\delta(t,z_\delta)=\int_\Omega \sigma_\delta(t){:}\dot{\mathbf{e}}_\mathrm{D}(t)\,\mathrm{d}x \text{ with}\\&\sigma_\delta(t,x)=\partial_e W(x,\mathbf{e}_\mathrm{D}(t,x)+\mathbf{e}(u_\delta(t))(x),z_\delta(t,x)).\end{aligned}$$

The main observation is that the stress-control assumption (1e) and the usual energy a priori estimates provide bounds for σ_δ in $\mathrm{L}^{p/(p-1)}(\Omega,\mathbf{E}_d)$ that are independent of $\delta>0$.

The essential idea to make the limit tractable is to introduce an auxiliary functional in which it is possible to keep control over the Γ-limit. Denote by $\mathbb{E}=\mathrm{L}^p(\Omega;\mathbf{E}_d)$ the strain space, and for $(e,z)\in\mathbb{E}\times\mathcal{Z}$ let

$$\begin{aligned}&\mathcal{H}_\delta(e,z)=\mathcal{V}_\delta(e,z)+\mathcal{G}(z) \text{ with}\\&\mathcal{V}_\delta(e,z)=\min\big\{\int_\Omega W_\delta(x,e{+}\mathbf{e}(u),z)\,\mathrm{d}x \mid u\in\mathcal{F}\big\}.\end{aligned} \tag{4}$$

In fact, the functional $\mathcal{V}_\delta$ should not be considered as a functional on $\mathbb{E}$ but rather on $\mathbb{B}=\{\,u|_{\partial\Omega}\mid u\in\mathcal{F}\,\}$, since all the other information is minimized out. Moreover, for fixed $z\in\mathcal{Z}$, the mapping $e\mapsto\mathcal{V}_\delta(e,z)$ is convex and differentiable with

$$\mathrm{D}_e\mathcal{V}_\delta(e,z)=\partial_e W(x,e{+}\mathbf{e}(V(e,z)),z)\in\mathbb{E}^*=\mathrm{L}^{p/(p-1)}(\Omega;\mathbf{E}_d),$$

where $V(e,z)\in\mathcal{F}$ is the unique minimizer in (4). This shows that $\sigma=\mathrm{D}_e\mathcal{V}_\delta(e,z)$ is in fact an equilibrium stress, and thus satisfies $\operatorname{div}\sigma=0$ in Ω and $\sigma\,\nu=0$ on $\partial\Omega\backslash\Gamma_\mathrm{D}$.

The importance of the functional $\mathcal{V}_\delta$ is that on the one hand it is possible to do the Γ-limit for $\delta\to0$ and keep some of the main features and that on the other hand, by construction the reduced functional $\mathcal{I}_\delta$ and its partial derivative with respect to t can be easily expressed:

$$\mathcal{I}_\delta(t,z)=\mathcal{V}_\delta(\mathbf{e}_\mathrm{D}(t),z){+}\mathcal{G}(z) \text{ and } \partial_t\mathcal{I}_\delta(t,z)=\langle\mathrm{D}_e\mathcal{V}_\delta(\mathbf{e}_\mathrm{D}(t),z),\dot{\mathbf{e}}_\mathrm{D}(t)\rangle.$$

Thus, we have found a way to express the energies in terms of the damage alone and we still have control over the equilibrium stresses $\mathrm{D}_e\mathcal{V}_\delta(\mathbf{e}_\mathrm{D}(t),z)$ that are needed to control the power generated by the boundary data $u_\mathrm{D}(t)$.

4 Existence for the complete-damage problem

A functional $\mathfrak{I}(t,\cdot):\mathcal{Z}\to\mathbb{R}$ is called the Γ-limit of $(\mathcal{I}_\delta(t,\cdot))_\delta$ if

$$\begin{aligned}&(\Gamma1)\quad z_\delta\rightharpoonup z \text{ in } \mathcal{Z} \quad\Longrightarrow\quad \mathfrak{I}(t,z)\le\liminf_{\delta\to0}\mathcal{I}_\delta(t,z_\delta),\\&(\Gamma2)\quad \forall\,z\in\mathcal{Z}\;\exists\,(z_\delta)_\delta:\; z_\delta\rightharpoonup z \text{ in } \mathcal{Z} \text{ and } \mathcal{I}_\delta(t,z_\delta)\to\mathfrak{I}(t,z).\end{aligned}$$

We note that Γ-convergence is quite different from pointwise convergence, see Example **4**.2. Moreover, while each $\mathcal{I}_\delta$ was strongly continuous, this is not true for $\mathfrak{I}(t,\cdot)$; only the important weak lower semicontinuity is maintained (as for all Γ-limits).

The main difficulty is to control the temporal smoothness of $\mathfrak{I}$, or more precisely to show that the following implication holds

$$\left.\begin{array}{c} z_\delta \rightharpoonup z \text{ in } \mathcal{Z} \\ \mathcal{I}_\delta(t,z_\delta) \to \mathfrak{I}(t,z) \end{array}\right\} \implies \partial_t \mathcal{I}_\delta(t,z_\delta) \to \partial_t \mathfrak{I}(t,z),$$

cf. condition (2.9) in [MRS08]. To provide this result we use the functional $\mathcal{V}_\delta$, since its Γ-limit can be studied more easily. The following result is a direct generalization of [BMR09, Prop. 2.10].

Proposition 4.1. *Let* (1) *hold with $p > 1$ and $r > d$. On $\mathbb{E} \times \mathcal{Z}$ define*

$$\mathfrak{V}(e,z) = \lim_{\varepsilon\to 0^+} \Big(\lim_{\delta\to 0^+} \mathcal{V}_\delta\big(e, \max\{z-\varepsilon, 0\}\big)\Big).$$

Then, $\mathfrak{V}$ satisfies

$$\exists\, C>0\ \forall\,(e,z)\in\mathbb{E}\times\mathcal{Z}: \quad -C \le \mathfrak{V}(e,t) \le C + C\|e\|_{\mathbb{E}}^p, \tag{5a}$$

$$\forall\, z\in\mathcal{Z}: \quad \mathfrak{V}(\cdot,z) \text{ is convex on } \mathbb{E}, \tag{5b}$$

$$\text{if } W(x,\cdot,z) \text{ is quadratic, then } \mathfrak{V}(\cdot,z) \text{ is quadratic.} \tag{5c}$$

Moreover, we have $\mathfrak{I}(t,z) = \mathfrak{V}(\mathbf{e}_D(t),z) + \mathcal{G}(z)$.

The proof relies on the compact embedding of $\mathrm{W}^{1,r}(\Omega)$ into $\mathrm{C}^0(\overline{\Omega})$ and uses essentially the monotonicity properties of the mapping $(\varepsilon,\delta) \mapsto \mathcal{V}_\delta\big(e,\max\{z-\varepsilon,0\}\big)$: it is non-increasing in ε because of (1d) and it is nondecreasing in δ because of the definition of W_δ in (3). Thus, the limit $\mathfrak{V}(e,z)$ always exists as a pointwise limit in δ and then in ε. Moreover, for each fixed z the convexity in e is preserved by pointwise convergence. The following example, which is inspired by [BoV88, Ex. 3] and further discussed in [BMR09], shows that in general $\mathfrak{V}$ is strictly smaller than $\mathcal{V}_0(e,z) = \lim_{\delta\to 0^+} \mathcal{V}_\delta(e,z)$.

Example 4.2. *Consider $\Omega = \,]-1,1[$ and the energy*

$$\mathcal{I}_\delta(t,z) = \int_\Omega \frac{\delta+z}{2} (\mathbf{e}_D(t) + u')^2 \,\mathrm{d}x + \mathcal{G}(z).$$

Then, $\mathcal{V}_\delta(e,z) = \big(\int_\Omega e \,\mathrm{d}x\big)^2 / \int_\Omega \frac{2}{\delta+z}\,\mathrm{d}x$. Clearly, the pointwise limit $\mathcal{V}_0$ is obtained by letting $\delta = 0$. However, the Γ-limit $\mathfrak{V}(e,\cdot)$ in $\mathrm{W}^{1,r}(\Omega)$ satisfies

$$\mathfrak{V}(e,z) = \mathcal{V}_0(e,z) \text{ for } \min z > 0 \quad \text{and } \mathfrak{V}(e,z) = 0 \text{ for } \min z = 0.$$

For $\alpha \in \,](r-1)/r, 1[$ we let $z_\alpha(x) = |x|^\alpha$, then $z_\alpha \in \mathcal{Z}$ and $0 = \mathfrak{V}(e,z) < \mathcal{V}_0(e,z) = (1-\alpha)\big(\int_\Omega e\,\mathrm{d}x\big)^2/4$.

The formula for $\mathfrak{I}$ allows us to study the question whether the power exists. For this, we use that convex functions have one-sided Gateaux derivatives in all points:

$$\begin{aligned} \delta_e \mathfrak{V}(e,z;\widehat{e}) &= \lim_{h\to 0^+} \frac{1}{h}\Big(\mathfrak{V}(e+h\widehat{e},z) - \mathfrak{V}(e,z)\Big) \\ &= \sup\big\{\, \langle \sigma, \widehat{e}\rangle \mid \sigma \in \partial_e^{\mathrm{sub}} \mathfrak{V}(e,z) \,\big\}, \end{aligned} \tag{6}$$

where $\partial_e^{\rm sub}\mathfrak{V}(e,z) \subset \mathbb{E}^*$ denotes the subdifferential of the convex function $\mathfrak{V}(\cdot,z)$. Using $\mathbf{e}_{\rm D} \in {\rm C}^1([0,T];\mathbb{E})$ we find that the left and right partial derivatives $\partial_t^{\pm}\mathfrak{I}(t,z) = \lim_{h\to 0^+} \frac{\pm 1}{h}\big(\mathfrak{I}(t{\pm}h,z) - \mathfrak{I}(t,z)\big)$ with respect to t of $\mathfrak{I}$ exists. We have the relations

$$\partial_t^-\mathfrak{I}(t,z) = -\delta_e\mathfrak{V}(t,\mathbf{e}_{\rm D}(t);-\dot{\mathbf{e}}_{\rm D}(t)) \le \delta_e\mathfrak{V}(t,\mathbf{e}_{\rm D}(t);\dot{\mathbf{e}}_{\rm D}(t)) = \partial_t^+\mathfrak{I}(t,z).$$

Definition 4.3. *Let $z:[0,T]\to\mathcal{Z}$ satisfy (S) in* (2) *for all $t\in[0,T]$. Then, z is called a* generalized energetic solution *of the rate-independent energetic system $(\mathcal{Z},\mathfrak{I},\mathcal{D})$, if there exists $p:[0,T]\to\mathbb{R}$ such that $p(\tau)\in\partial_\tau^{Cl}\mathfrak{I}(\tau,z(\tau))$ a.e. in $[0,T]$ and for all $t\in[0,T]$ we have*

$$\mathfrak{I}(t,z(t)) + {\rm Diss}_{\mathcal{D}}(z,[0,t]) = \mathfrak{I}(0,z(0)) + \int_0^t p(\tau)\,{\rm d}\tau. \tag{7}$$

Now a slight generalization of the abstract existence theory for rate-independent systems gives the following. Note that we construct weak energetic solutions for $(\mathcal{Z},\mathfrak{I},\mathcal{D})$ directly directly, without reference to the solutions z_δ for $(\mathcal{Z},\mathfrak{I}_\delta,\mathcal{D})$.

Theorem 4.4. *For all stable $z^0\in\mathcal{Z}$ there exists a generalized energetic solution for $(\mathcal{Z},\mathfrak{I},\mathcal{D})$.*

Proof. The existence theory follows the usual steps in the abstract theory for rate-independent processes [Mie05, FrM06] via incremental minimization, uniform a priori estimates and Helly's selection principle. This part and the proof of the stability of the limit process work as in [BMR09].

For the upper energy estimate we obtain, by setting $A(t) = \mathfrak{I}(t,z(t)) + {\rm Diss}_{\mathcal{D}}(z,[0,t])$,

$$A(s) - A(r) \le \int_r^s p^{\max}(t)\,{\rm d}t \text{ with } p^{\max}(t) = \max\partial_t^{\rm Cl}\mathfrak{I}(t,z(t)).$$

With a slight generalization of [Mie05, Prop. 5.7] we see that stability of the limit process z implies the lower bound $A(s)-A(r) \ge \int_r^s p^{\min}(t)\,{\rm d}t$ with $p^{\min}(t) = \min\partial_t^{\rm Cl}\mathfrak{I}(t,z(t))$.

Thus, we conclude that A is absolutely continuous and satisfies $p^{\min}(t) \le A'(t) \le p^{\max}(t)$. Hence, setting $p(t) = A'(t)$ the proof is complete. ☐

In the following example we show that the notion of generlaized energetic solution, which involves the weakened energy balance (7) with the Clarke differential, is really necessary in cases where the one-sided partial derivatives satisfy $\partial_t^-\mathfrak{I}(t,z) < \partial_t^+\mathfrak{I}(t,z)$ at some points. In particular, it is not possible to make an a priori choice like $p(t) = \max\{\partial_t^{\rm Cl}\mathfrak{I}(t,z(t))\}$, which worked in [KZM10, MiR09], since there $\partial_t^-\mathfrak{I}(t,z) \ge \partial_t^+\mathfrak{I}(t,z)$ holds.

Example 4.5. *This example has a smooth energy $\mathfrak{I}_\delta$ such that $\partial_t\mathfrak{I}_\delta$ exists, while in the limit $\mathfrak{I}$ is only Lipschitz in t. We let $\mathcal{Z}=\mathbb{R}$ and $\mathcal{D}(z,\widetilde{z}) = |\widetilde{z}{-}z|$. The energy functional reads*

$$\mathfrak{I}_\delta(t,z) = H_\delta\big(z{-}\alpha(t)\big) \quad \textit{and} \quad \mathfrak{I}(t,z) = 2|z{-}\alpha(t)|,$$

where $\alpha \in \mathrm{C}^1([0,T])$ *is given and* $H_\delta(u) = 2u^2/\sqrt{\delta^2{+}u^2}$. *For the partial derivatives with respect to time we have*

$$\partial_t \mathcal{I}_\delta(t,z) = -H_\delta'(z{-}\alpha(t))\dot\alpha(t) \text{ and } \partial_t^{Cl}\mathfrak{I}(t,z) = -2\,\mathrm{Sign}(z{-}\alpha(t))|\dot\alpha(t)|.$$

Since $\mathcal{I}_\delta(t,\cdot)$ *is smooth and strictly convex, the energetic solutions for* $(\mathbb{R},\mathcal{I}_\delta,\mathcal{D})$ *are exactly the solutions of the doubly nonlinear equation [MiT04]*

$$0 \in \mathrm{Sign}(\dot z(t)) + H_\delta'(z(t){-}\alpha(t)).$$

For $\delta > 0$ *the system is smooth, while for* $\delta = 0$ *we have* $H_0(u) = 2|u|$ *and set* $\mathfrak{I}(t,z) = H_0(z{-}\alpha(t))$.

Consider the special case $\alpha(t) = t$ *and* $z_\delta(0) = 0$. *If* β_δ *is the unique solution of* $H_\delta'(\beta_\delta) = 1$, *then the unique energetic solution is* $z_\delta(t) = \max\{0, t{-}\beta_\delta\}$. *Using* $0 < \beta_\delta \to 0$ *we find the limit solution* $z(t) = t = \lim_{\delta\to 0} z_\delta(t)$. *It is a generalized energetic solution in the sense of Definition* **4**.*3 by using* $p(t) = 1 \in [-2,2] = \partial_t^{Cl}\mathfrak{I}(t,t)$.

5 Γ-convergence for $\delta \to 0$

Here we discuss the Γ-limit for the solutions z_δ of the rate-independent energetic system $(\mathcal{Z},\mathcal{I}_\delta,\mathcal{D})$. First note that the a priori estimates give the boundedness of the family $(z_\delta)_\delta$ in $\mathrm{BV}([0,T],\mathrm{L}^1(\Omega)) \cap \mathrm{L}^\infty([0,T],\mathrm{W}^{1,r}(\Omega))$, and hence Helly's selection principle allows us to extract a subsequence $(z_{\delta_k})_{k\in\mathbb{N}}$ which converges pointwise on $[0,T]$ to a limit $z : [0,T] \to \mathcal{Z}$ satisfying the same bound, i.e., $z_\delta(t) \rightharpoonup z(t)$ in $\mathcal{Z}$.

To conclude that z is a generalized energetic solution for $(\mathcal{Z},\mathfrak{I},\mathcal{D})$ it is sufficient to check two compatibility conditions, namely *conditioned continuous convergence of the power*, cf. [MRS08, (2.9)], and *conditioned upper semicontinuity of stable sets*, cf. [MRS08, (2.11)]. The latter condition is purely static and it is not difficult to generalize it to the present case. As in [BMR09] we obtain the energy convergence $\mathcal{I}_\delta(t,z_\delta(t)) \to \mathfrak{I}(t,z(t))$, which in turn implies strong convergence $\|z_\delta(t){-}z(t)\|_{\mathrm{W}^{1,r}} \to 0$.

The conditional continuous convergence of the power would be satisfied if the following conjecture would be true.

Conjecture 5.1. *Assume that* z_{δ_j} *is stable for* $(\mathcal{Z},\mathcal{I}_{\delta_j},\mathcal{D})$ *at time* t, $z_{\delta_j} \rightharpoonup z$, $\mathcal{I}_{\delta_j}(t,z_{\delta_j}) \to \mathfrak{I}(t,z)$, *and* $\sigma_{\delta_j} = \mathrm{D}_e\mathcal{V}_{\delta_j}(\mathbf{e}_D(t),z_{\delta_j}) \rightharpoonup \sigma_*$ *in* $\mathbb{E}^*$, *then* $\sigma_* \in \partial_e^{sub}\mathfrak{V}(\mathbf{e}_D(t),z)$.

The conjecture holds [BMR09] under the assumption that $W(x,e,z)$ is quadratic in e. The relevant consequence is obtained via (6):

$$\partial_t^- \mathfrak{I}(t,z) \le \liminf_{\delta\to 0} \partial_t\mathcal{I}_\delta(t,z_\delta) \le \limsup_{\delta\to 0} \partial_t\mathcal{I}_\delta(t,z_\delta) \le \partial_t^+\mathfrak{I}(t,z). \tag{8}$$

Combining this estimate with the abstract Γ-convergence for rate-independent systems [MRS08] and the existence theory for complete damage [BMR09] it is possible to obtain the following convergence result.

Theorem 5.2. *Assume that the (yet unproved) estimate* (8) *holds. If* $(z_\delta)_{\delta\in(0,1)}$ *is a family of solutions to* $(\mathcal{Z}, \mathcal{I}_\delta, \mathcal{D})$ *satisfying*

$$z_\delta(0) \rightharpoonup z^0 \ \text{in}\ \mathrm{W}^{1,r}(\Omega) \quad \text{and} \quad \mathcal{I}_\delta(0, z_\delta(0)) \to \mathfrak{I}(0, z^0),$$

then there exist a subsequence $(z_{\delta_j})_{j\in\mathbb{N}}$ *and a generalized energetic solution* $z : [0, T] \to \mathcal{Z}$ *for* $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$ *with* $z(0) = z^0$ *such that for all* $t \in [0, T]$

$$z_{\delta_j}(t) \to z(t) \ \text{in}\ \mathrm{W}^{1,r}(\Omega), \quad \mathcal{I}_\delta(t, z_\delta(t)) \to \mathfrak{I}(t, z(t)),$$
$$\mathrm{Diss}_{\mathcal{D}}(z_\delta, [0, t]) \to \mathrm{Diss}_{\mathcal{D}}(z, [0, t]).$$

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A vanishing viscosity approach to rate-independent modelling in metric spaces

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Abstract

We present a new variational formulation of rate-independent evolutionary problems in metric spaces, which arises in the limit of a vanishing viscosity approximation.

Keywords: Rate-independent evolutions, vanishing viscosity, analysis in metric spaces.

1 Introduction

Rate-independent problems arise in a variety of applicative contexts, among which elastoplasticity, damage, the quasistatic evolution of fractures, shape memory alloys, delamination and ferromagnetism, see [3] and the references therein. In several situations, the evolution of rate-independent systems is described by the doubly nonlinear equation

$$\partial\Psi(u'(t)) + \mathrm{D}_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \;\; \text{for a.a. } t \in (0, T)\,, \tag{1}$$

where B is a (separable) Banach space, $\Psi : B \to \mathbb{R}^+$ a lower semicontinuous and convex *dissipation* functional (for simplicity, here we omit its dependence on the state variable u), fulfilling $\Psi(\lambda v) = \lambda\Psi(v)$ for all $\lambda \geq 0$ and $v \in B$; the symbol ∂ denotes its subdifferential, and, hereafter, we shall assume the *energy potential* $\mathcal{E} : [0,T] \times B \to (-\infty, +\infty]$ (possibly accounting for an external loading through the dependence on the variable t) to be smooth with respect to t. Indeed, (1) renders a balance between a potential restoring force $-\mathrm{D}_u\mathcal{E}(t,u)$ and a frictional force $f \in \partial\Psi(u')$. Since $\partial\Psi$ is homogeneous of degree 0, (1) is invariant for time rescalings, meaning that the process under consideration is insensitive to changes in the time scale. Such a feature is typical of mechanical systems driven by an external loading set on a time scale much slower than the system internal time scale, which is thus neglected. This corresponds to taking the vanishing viscosity limit of systems with a viscous, rate-dependent dissipation.

When B is a reflexive Banach space and $\mathcal{E}(t,\cdot)$ is uniformly convex and smooth, the Cauchy problem for (1) has a unique solution $u \in \mathrm{W}^{1,\infty}([0,T];B)$, see [3, Sec. 3]. On the other hand, the relevant energy functionals in many applications are neither smooth, nor, in general, convex. In such cases, existence can still be proved by passing to the limit in a time-discretization scheme via energy a priori estimates and compactness/lower semicontinuity arguments, but solutions are in general only BV with respect to time, hence they may jump. Furthermore, the natural ambient space for problems in, e.g., shape-memory alloys, quasistatic crack growth, finite-strain elastoplasticity, need not be reflexive, and may even lack a linear/differential structure. These considerations show that the subdifferential formulation (1), involving both the pointwise derivative of u and the (Gâteaux) differential of $\mathcal{E}(t,\cdot)$ with respect to u, is often not adequate for rate-independent modelling. Hence, a *derivative-free* formulation (leading to the notion of *energetic solution* of the rate-independent system (1)) has been proposed, combining an energy balance with a *global stability* inequality, see [3, Sec. 2.1]. However, for nonconvex energies the latter condition may force solutions to jump "too early" to avoid energy losses.

The purpose of this note is to present a novel formulation of rate-independent evolutions which, on the one hand, is based on a *local*, rather than a global, stability condition, and does not enforce premature or spurious jumps, at the same time providing a description of the jump paths. Furthermore, we aim to develop our analysis in a general setting, and for nonsmooth, nonconvex energies. Indeed, we shall work in a complete metric space and, according to physical intuition, derive our new solution notion by passing to the limit as $\varepsilon \searrow 0$ in the viscous problem

$$\varepsilon J(u'(t)) + \partial\Psi(u'(t)) + \mathrm{D}_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad \text{for a.a. } t \in (0,T) \tag{2}$$

($J : B \to B'$ being the duality operator). In doing so, we shall follow the approach of [2], which we briefly sketch.

The vanishing viscosity analysis by Efendiev & Mielke. In [2], the case $B = \mathbb{R}^N$, $\Psi(v) = K\|v\|$ for all $v \in \mathbb{R}^N$ and some $K > 0$, and $\mathcal{E} \in \mathrm{C}^1([0,T]\times\mathbb{R}^N)$ was considered. The key idea in [2] is that jumps are in fact viscous transitions of the system between two metastable states, which are very fast with respect to the slow external time scale. In order to capture the viscous transition path at jumps, the authors go over to the trajectory phase space, reparametrize the sequence of solutions $\{u_\varepsilon\}_\varepsilon$ to (2) by their arclength τ_ε, and study the limiting behavior as $\varepsilon \searrow 0$ of the sequence $\{(\widehat{t}_\varepsilon := \tau_\varepsilon^{-1}, \widehat{u}_\varepsilon := u_\varepsilon(\widehat{t}_\varepsilon))\}_\varepsilon$. With some calculations, one sees that, setting

$$\widehat{\Psi}_\varepsilon(v) := \Psi(v) + \varepsilon(-\|v\| - \log(1 - \|v\|)) + I_{[0,1)}(\|v\|) \quad \text{for all } v \in \mathbb{R}^N$$

(where $I_{[0,1)}$ is the indicator function of $[0,1)$), the pair $(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)$ fulfils for every $\varepsilon > 0$

$$\begin{aligned} &\partial\widehat{\Psi}_\varepsilon(\widehat{u}_\varepsilon'(\tau)) + \mathrm{D}_u\mathcal{E}(\widehat{t}_\varepsilon(\tau), \widehat{u}_\varepsilon(\tau)) \ni 0 && \text{for a.a. } \tau \in (0, \widehat{T}_\varepsilon)\,, \\ &\widehat{t}_\varepsilon'(\tau) + \|\widehat{u}_\varepsilon'(\tau)\| = 1 && \text{for a.a. } \tau \in (0, \widehat{T}_\varepsilon)\,, \end{aligned}$$

with $\widehat{T}_\varepsilon = \tau_\varepsilon(T)$. Hence, in [2] it was proved that, up to a subsequence, $\{(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)\}_\varepsilon$ converges to a pair $(\widehat{t}, \widehat{u}) \in \mathrm{AC}([0, \widehat{T}]; [0,T] \times X)$ satisfying

$$\begin{aligned} &\partial\widehat{\Psi}(\widehat{u}'(\tau)) + \mathrm{D}_u\mathcal{E}(\widehat{t}(\tau), \widehat{u}(\tau)) \ni 0 && \text{for a.a. } \tau \in (0, \widehat{T})\,, \\ &\widehat{t}'(\tau) + \|\widehat{u}'(\tau)\| = 1 && \text{for a.a. } \tau \in (0, \widehat{T})\,, \end{aligned} \tag{3}$$

and $\widehat{t}(0) = 0$, $\widehat{t}(\widehat{T}) = T$, where $\widehat{T} = \lim_{\varepsilon\searrow 0} \widehat{T}_\varepsilon$ and $\widehat{\Psi}_\varepsilon$ Mosco-converges to the function $\widehat{\Psi}(v) := \Psi(v) + I_{[0,1]}(\|v\|)$ for all $v \in \mathbb{R}^N$. The limiting problem (3) encompasses three regimes, which completely describe the evolution of the rate-independent system:

sticking corresponding to $\|\widehat{u}'\| = 0$ ($\Leftrightarrow \widehat{t}' = 1$);

sliding (or *rate-independent motion*), occurring when $0 < \widehat{t}', \|\widehat{u}'\| < 1$: in this case, (3) may be reparametrized to yield (1);

viscous slip when $\|\widehat{t}'\| = 0$: then, the system switches to a rate-dependent behavior encoded by (3), which yields a gradient flow at the fixed process time $\widehat{t}$.

2 The metric formulation of doubly nonlinear evolution equations

Before revisiting the vanishing viscosity approach of [2] in the framework of

a *complete metric space* (X, d),

we show how doubly nonlinear equations of the type (1) (where for the moment Ψ is no longer 1-homogeneous) may be reformulated in the absence of a linear/differentiable structure on X. We note that (1) is equivalent for a.e. $t \in (0,T)$ to

$$\Psi(u'(t)) + \Psi^*(-\mathrm{D}_u\mathcal{E}(t, u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t\mathcal{E}(t, u(t)) = 0\,, \tag{4}$$

where Ψ^* is the convex conjugate of Ψ and we have combined the convex analysis property

$$\Psi(v) + \Psi^*(\sigma) = \langle \sigma, v \rangle \;\Leftrightarrow\; \sigma \in \partial\Psi(v)$$

with the (formal) chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = \langle \mathrm{D}_u\mathcal{E}(t,u(t)), u'(t)\rangle.$$

Now, choosing $\Psi(v) = \|v\|^p$, $1 < p < \infty$, we see that (4) highlights the role of the modulus of the derivatives u' and $-\mathrm{D}_u\mathcal{E}$, rather than of the derivatives themselves.

Thus, (4) somehow points in the direction of the appropriate formulation of (1) in the metric space (X,d). There, one in fact disposes of surrogates of the modulus of derivatives, in the realm of E. De Giorgi's theory of *Curves of Maximal Slope*, (i.e. of gradient flow equations in metric spaces, see [1] and the references therein). Indeed, for every absolutely continuous curve $u : (0,T) \to X$,

$$\text{the limit } \lim_{h\to 0}\frac{d(u(t+h),u(t))}{|h|} =: |u'|(t) \text{ exists for a.a. } t\in(0,T),$$

defining the *metric derivative* of u (which "replaces" the norm of the pointwise derivative $\|u'\|$). In the same way, given a functional $\mathcal{E} : [0,T]\times X \to (-\infty,+\infty]$, the *local slope* of $\mathcal{E}$ at $u \in \mathrm{dom}(\mathcal{E}(t,\cdot))$

$$|\partial\mathcal{E}|(t,u) := \limsup_{z\to u}\frac{(\mathcal{E}(t,u)-\mathcal{E}(t,z))^+}{d(u,z)} \tag{5}$$

surrogates $\|-\mathrm{D}_u\mathcal{E}(t,u(t))\|$. In this framework, the usual chain rule identity is substituted by a *chain rule inequality*, viz.: For all $u \in \mathrm{AC}([0,T];X)$

$$\begin{gathered}\text{the map } t\mapsto\mathcal{E}(t,u(t)) \text{ is absolutely continuous on } [0,T], \text{ and}\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) \ge -|u'|(t)\cdot|\partial\mathcal{E}|(t,u(t)) \quad \text{for a.a. } t\in(0,T).\end{gathered} \tag{6}$$

With the help of these concepts, in [4] we have proposed this definition: given a (l.s.c., convex) function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a curve $q \in \mathrm{AC}([0,T];X)$ fulfils the ψ-gradient system associated with a functional $\mathcal{E} : [0,T]\times X \to (-\infty,+\infty]$ if for a.a. $t\in(0,T)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = -\psi(|u'|(t)) - \psi^*\big(|\partial\mathcal{E}|(t,u(t))\big), \tag{7}$$

(which is in fact formally analogous to (4)). In [4] it is shown that, under suitable assumptions, the Cauchy problem for (7) has at least a solution, and that (7) is a reformulation of (1) when X is a Banach space with the Radon-Nikodým property.

3 Parametrized metric solutions of rate-independent systems

Setting $\psi_\varepsilon(x) = x + \frac{\varepsilon}{2}x^2$ for all $x\in\mathbb{R}^+$, the metric formulation of (2) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u_\varepsilon(t)) - \partial_t\mathcal{E}(t,u(t)) = -\psi_\varepsilon(|u_\varepsilon'|(t)) - \psi_\varepsilon^*\big(|\partial\mathcal{E}|(t,u_\varepsilon(t))\big). \tag{8}$$

for a.a. $t \in (0,T)$. In [5, 7] we adapt the techniques of [2] to the metric framework: we rescale u_ε by the arclength $\tau_\varepsilon(t) := t + \int_0^t |u_\varepsilon'|(s)\mathrm{d}s$, $t \in [0,T]$, and pass to the limit as $\varepsilon \searrow 0$ in the (rescaled) metric formulation fulfilled by the pair $(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)$. Under suitable assumptions on $\mathcal{E}$ and the chain rule (6), combining a priori estimates with compactness/lower semicontinuity arguments, we indeed prove that, up to a subsequence, $\{(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)\}$ converges to a pair $(\widehat{t}, \widehat{u}) \in \mathrm{AC}([0,\widehat{T}];[0,T]\times X)$ (with $\widehat{T} = \lim_\varepsilon \tau_\varepsilon(T)$), fulfilling $\widehat{t}(0)=0$, $\widehat{t}(\widehat{T}) = T$ and

$$\left.\begin{array}{l}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{E}(\widehat{t}(\tau),\widehat{u}(\tau)) - \partial_t\mathcal{E}(\widehat{t}(\tau),\widehat{u}(\tau))\widehat{t}'(\tau) = -\widehat{\psi}(|\widehat{u}'|(\tau)) - \widehat{\psi}^*\big(|\partial\mathcal{E}|(\widehat{t}(\tau),\widehat{u}(\tau))\big),\\ \widehat{t}'(\tau) + |\widehat{u}'|(\tau) = 1\end{array}\right\} \tag{9}$$

for a.a. $\tau \in (0,\widehat{T})$, where, in accordance with (3), $\widehat{\psi}(x) = x + I_{[0,1]}(x)$ for all $x \in \mathbb{R}^+$. Let us point out that, in view of the chain rule (6) and elementary convex analysis, the first of (9) can be decoupled as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{E}(\widehat{t}(\tau),\widehat{u}(\tau)) - \partial_t\mathcal{E}(\widehat{t}(\tau),\widehat{u}(\tau))\widehat{t}'(\tau) = -|\widehat{u}'|(\tau)\cdot|\partial\mathcal{E}|(\widehat{t}(\tau),\widehat{u}(\tau)) \quad \text{for a.a. } \tau \in (0,\widehat{T}), \tag{10}$$

$$|\partial\mathcal{E}|(\widehat{t}(\tau),\widehat{u}(\tau)) \in \partial\widehat{\psi}(|\widehat{u}'|(\tau)) \quad \text{for a.a. } \tau \in (0,\widehat{T}). \tag{11}$$

Taking into account that $\partial\widehat{\psi}(0) = (-\infty,1]$, $\partial\widehat{\psi}(v) = \{1\}$ for $v \in (0,1)$, and $\partial\widehat{\psi}(1) = [1,+\infty)$, also in view of the second of (9) we rephrase (11) as

$$\left.\begin{array}{lll}|\widehat{u}'|(s) = 1\ (\Leftrightarrow \widehat{t}'(s) = 0) & \Rightarrow & |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1,\\ |\widehat{u}'|(s) \in (0,1)\ (\Leftrightarrow \widehat{t}'(s) \in (0,1)) & \Rightarrow & |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1,\\ |\widehat{u}'|(s) = 0\ (\Leftrightarrow \widehat{t}'(s) = 1) & \Rightarrow & |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1,\end{array}\right\} \tag{12}$$

which highlights three regimes: *sticking*, corresponding to $|\widehat{u}'| = 0$, *sliding*, occurring when $\widehat{t}'\cdot|\widehat{u}'| > 0$, and *viscous slip*, at frozen process time (i.e. $\widehat{t}' = 0$). The notion of rate-independent metric evolution we propose retains (10) and (12), replacing the second of (9) with a general "non-degeneracy" condition.

Definition 3.1. *A curve* $(\widehat{t},\widehat{q}) \in \mathrm{AC}([0,\widehat{T}];[0,T]\times X)$ *is a* parametrized metric solution *of the rate-independent system* $(X,d,\mathcal{E})$ *if* $\widehat{t} : [0,\widehat{T}] \to [0,T]$ *is non-decreasing, the energy identity* (10) *and the differential conditions* (12) *hold, and*

$$\widehat{t}'(\tau) + |\widehat{u}'|(\tau) > 0 \qquad \textit{for a.a. } \tau \in (0,\widehat{T}). \tag{13}$$

We refer to [5] for a thorough discussion of this concept in a specific metric setting (viz., a finite-dimensional smooth manifold endowed with a Finsler distance), and comparison with *energetic solutions* and other solution notions for rate-independent problems. Here, we just point out that (10), (12), and (13) (unlike the second of (9)), are invariant with respect to strictly increasing reparametrizations: hence, the notion of parametrized metric solution is truly rate-independent. As we have seen, existence of parametrized metric solutions can be obtained by passing to the limit in the arclength-rescaled viscous approximation (8). In the forthcoming [7], we shall prove this in a general metric setting. Therein, we shall also prove existence of parametrized metric solutions, through approximation by time discretization and solution of incremental (*local*) minimization problems (cf. also the results in [6], in a Banach context). In doing so, we shall exploit the techniques of De Giorgi's theory of *Minimizing Movements* for the approximation of *Curves of Maximal Slope*, see [1, Chap. III].

Finally, let us gain further insight into the evolution (10) and (12).

Sliding : if $\widehat{t}'(\bar\tau), |\widehat{u}'|(\bar\tau) > 0$ at $\bar\tau \in (0, \widehat{T})$, in some neighborhood $(\bar\tau - \delta, \bar\tau + \delta)$ the system evolution is described by

$$\begin{cases} \mathcal{E}(\widehat{t}(\tau_2), \widehat{u}(\tau_2)) - \mathcal{E}(\widehat{t}(\tau_1), \widehat{u}(\tau_1)) = \int_{\tau_1}^{\tau_2} \big(\partial_t \mathcal{E}(\widehat{t}(\tau), \widehat{u}(\tau))\, \widehat{t}'(\tau) - |\widehat{u}'|(\tau)\big)\, \mathrm{d}\tau\,, \\ |\partial \mathcal{E}|\,(\widehat{t}(\tau), \widehat{u}(\tau)) = 1 \end{cases}$$

for a.a. $\tau \in (\bar\tau - \delta, \bar\tau + \delta)$, where the former relation, obtained by integrating (10) for all $\tau_1, \tau_2 \in (\bar\tau - \delta, \bar\tau + \delta)$, is an *energy balance* and the latter one a *local stability* condition, since the slope notion (5) has an intrinsically local character.

Viscous slip : if $\widehat{t}'(\bar\tau) = 0$ at $\bar\tau \in (0, \widehat{T})$, then $\widehat{t}(\tau) = \widehat{t}(\bar\tau)$ for $\tau \in (\bar\tau - \delta, \bar\tau + \delta)$, and

$$\begin{cases} \mathcal{E}(\widehat{t}(\bar\tau), \widehat{u}(\tau_2)) - \mathcal{E}(\widehat{t}(\bar\tau), \widehat{u}(\tau_1)) = -\int_{\tau_1}^{\tau_2} |\partial \mathcal{E}|\,(\widehat{t}(\bar\tau), \widehat{u}(\tau))\, |\widehat{u}'|(\tau)\, \mathrm{d}\tau\,, \\ |\partial \mathcal{E}|\,(\widehat{t}(\tau), \widehat{u}(\tau)) \geq 1, \quad |\widehat{u}'|(\tau) > 0 \end{cases}$$

for a.a. $\tau \in (\bar\tau - \delta, \bar\tau + \delta)$, where the energy identity for all $\tau_1, \tau_2 \in (\bar\tau - \delta, \bar\tau + \delta)$ provides a description of the system evolution along the jump path, in terms of a (*generalized*) gradient flow at the fixed process time $\widehat{t}(\bar\tau)$.

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Fold bifurcations and linear stability analysis in systems with Preisach hysteresis

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Abstract

We propose an algorithm of linear stability analysis for periodic solutions of operator-differential equations involving the time derivative of the output of the Preisach operator. The results are tested numerically by examples where a periodic solution undergoes the fold bifurcation.

Keywords: Operator-differential equation, Preisach operator, Stability analysis, Fold bifurcation, Hysteresis.

1 Introduction

We consider scalar operator-differential equations that contain the time derivative of the output of the Preisach operator P[1]. Such equations appear, for example, as models of the water flow through unsaturated soil exhibiting soil-moisture hysteresis in terrestrial hydrology (and, more generally, as models of other liquid flows through porous media). In particular, models

$$ax'(t) + (Px)'(t) = f(t, x(t)), \qquad a > 0, \tag{1}$$

where P describes the hysteresis relation between the matric potential and the water content in the soil matrix[1], have been proposed and studied in [2]. The function f in the balance equation (1) can have different form depending on the type and nature of flows present in the system[3].

The Cauchy problem for equation (1) has been studied in [5]; numerical algorithms for its solution have been proposed and implemented in [6]. The

[1]Systems involving equations of the type (1) are also used for modelling electronic circuits with ferromagnetic hysteresis in inductance elements, see e.g. [4].

Preisach operator introduces a special type of memory in the system: it remembers certain local extrema of the solution from the past (the so-called main extrema or shock values). This memory manifests itself, for example, through the jumps of the derivative of the solution at the moments when a past shock value is reached again.

In [7] , an algorithm for linear stability analysis of periodic solutions of equation (1) has been proposed. The results refer to the class of the periodic solutions that have exactly one maximum and one minimum on a period. In this letter we test the proposed algorithm numerically by applying it to systems where a periodic solution undergoes the fold bifurcation, i.e. a stable and an unstable solutions collide and annihilate at the bifurcation point. Then we propose and discuss a modification of the algorithm, which extends it to a more general class of periodic solutions with an arbitrary number of local extrema on the period. To explain this modification it suffices to consider the solutions with four local extrema. The main result is formulated and illustrated by a numerical example for this case.

We note that the variation of initial data of equation (1) includes the variation of the initial state of the Preisach nonlinearity in an infinite dimensional metric space without a good linear structure. However, we show that for a generic class of admissible perturbations of the initial data stability of a solution is determined by a finite dimensional linear system.

2 Preisach nonlinearity

We use the following class of the Preisach models (operators); a more general definition and the phenomenology can be found, for example, in [1]. The output of the model is a scalar continuous function defined by the formula

$$y(t) = P[\eta_0]x(t) := \iint_{\omega(t)} \mu(\alpha, \beta)\, d\alpha\, d\beta \tag{2}$$

where a nonnegative integrable function $\mu : \Pi \to \mathbb{R}$, called the Preisach measure density, is defined on the strip $\Pi = \{0 \leq \beta - \alpha \leq d\}$ of the (α, β) plane; the domain $\omega(t) \subset \Pi$ of integration changes in time; $\eta_0 = \eta_0(\xi)$ is the initial state. For each t, the domain $\omega(t)$ has the form $\omega(t) = \{(\alpha, \beta) \colon \alpha + \beta \leq 2x(t) + \eta(t; \beta - \alpha)\}$, where $x(t)$ is a scalar continuous input of the model and the function $\eta(t; \cdot) : [0, d] \to \mathbb{R}$, called the state of the model at the moment t, satisfies $\eta(\cdot; 0) = 0$, $|\eta(\cdot; \xi_1) - \eta(\cdot; \xi_2)| \leq |\xi_1 - \xi_2|$. The evolution of the state is determined by simple rules, see [1]. We use the standard distance in the state space, $\rho(\eta_0^1, \eta_0^2) = \max_{\xi \in [0,d]} |\eta_0^1(\xi) - \eta_0^2(\xi)|$.

3 Properties of solutions of equation (1)

A solution $x = x(t)$ of equation (1) with the Preisach operator $P = P[\eta_0]$ defined by (2) satisfies the equation everywhere; the function $x + aPx$ in the left hand side is continuously differentiable, although x' and $(Px)'$ can have jumps. Solutions depend continuously on the initial data $t_0, x(t_0) = x_0$ and η_0. Algorithms of numerical solution of equation (1) and linearisation of the evolution operator is based on the following two properties of solutions [5, 6].

i) Each solution of (1) decreases in the area of the (t, x)-plane where $f(t, x) < 0$, increases in the area where $f(t, x) > 0$ and has extrema on the lines $f(t, x) = 0$.

ii) Suppose that a solution $x(t)$ of is monotone on a segment $[t_1, t_2]$ and $\eta_1 = \eta_1(\xi)$ is the state of the Preisach operator at the moment t_1. Set

$$J_\beta(y, x) = \int_x^y \mu(x, \beta)d\beta, \qquad J_\alpha(y, x) = \int_y^x \mu(\alpha, x)d\alpha$$

and denote by $\alpha_{\eta_1}(\beta), \beta_{\eta_1}(\alpha)$ the left-continuous monotone functions, which are uniquely defined by the relations (where one excludes ξ)

$$\alpha = x(t_1) + (-\xi + \eta_1(\xi))/2, \qquad \beta = x(t_1) + (\xi + \eta_1(\xi))/2.$$

If $x(t)$ decreases, then almost everywhere on $[t_1, t_2]$

$$x' = f(t, x)/(a + J_\beta(\beta_{\eta_1}(x), x)); \tag{3}$$

if $x(t)$ increases, then almost everywhere

$$x' = f(t, x)/(a + J_\alpha(\alpha_{\eta_1}(x), x)). \tag{4}$$

These properties imply that a solution $x = x(t)$ of (1) satisfies ordinary differential equations (3), (4) on consecutive intervals $[t_k, t_{k+1}]$ where t_k are the moments when $x(t)$ crosses the lines $f(t, x) = 0$. We note that the jumps of the functions $\alpha_{\eta_1}, \beta_{\eta_1}$ generate the jumps of the derivative of x.

If $x_* = x_*(t)$ is a periodic solution of Eq. (1) and $\eta_*(t_0) = \eta_0^*$ is a corresponding initial state of the Preisach operator, then x_* is a periodic solution for each initial state η_0 from a certain class $\Xi = \Xi(x_*)$. We denote by $\Xi_\kappa(x_*)$ the subclass of the class $\Xi(x_*)$, which consists of the states η_0 satisfying the relation $|\eta_0(\xi) - \eta_0(r_*)| \le \kappa(\xi - r_*)$ for $\xi > r_*$ with a $\kappa < 1$ and $r_* := \max_{t>\tau} |x_*(t) - x_*(\tau)|$. Following [7], we say that a periodic solution x_* of (1) is asymptotically stable with respect to admissible perturbations of initial data if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x(t) - x_*(t)| < \varepsilon$ for all $t > t_0$ and $|x(t) - x_*(t)| \to 0$ whenever $|x(t_0) - x^*(t_0)| < \delta$ and the distance from the initial state $\eta(t_0)$ of the Preisach operator for the perturbed solution x from the set $\Xi_\kappa(x_*)$ is less than δ.

4 Stability of a simple periodic solution

Formulas (3), (4) can be used as a basis for linearisation of equation (1) and the linear stability analysis. From now on, we consider T-periodic in t smooth functions f. Assume that (1) has a T-periodic solution $x_*(t)$, which has one

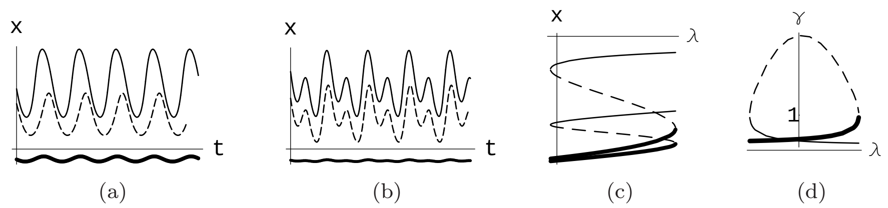

Figure 1: Periodic solutions with different number of extrema on a period: (a) 2 extrema; (b) 4 extrema. (c) Minimum and maximum values of periodic solutions with 2 extrema and (d) corresponding γ, depending on λ. The solid lines show stable solutions; the dashed lines are unstable solutions.

minimum and one maximum on a period: the minimum x_*^m is reached at a point t_1^* and the maximum x_*^M is reached at a point t_2^* with $t_0 < t_1^* < t_2^* < t_0 + T$. Also, assume that the right and left second derivatives of $x_*(t)$ are positive at the point t_1^* and negative at t_2^*. Set

$$\hat{f}(t,x,\alpha_0) := f(t,x)/(a + J_\alpha(\alpha_0,x)), \quad \widetilde{f}(t,x,\beta_0) := f(t,x)/(a + J_\beta(\beta_0,x)),$$

$a_1(t) = \hat{f}_x(t,x_*,x_*^m)$, $b_1(t) = \hat{f}_{\alpha_0}(t,x_*,x_*^m)$, $a_2(t) = \widetilde{f}_x(t,x_*,x_*^M)$, $b_2(t) = \widetilde{f}_{\beta_0}(t,x_*,x_*^M)$ and consider the solutions u, v, z_1, z_2 of the problems

$$u'(t) = a_2(t)u(t),\ u(t_0) = 1; \qquad v'(t) = a_2(t)v(t) + b_2(t),\ v(t_0) = 0;$$
$$z_1'(t) = a_1(t)z_1(t) + b_1(t),\ z_1(t_1^*) = 1;\ z_2'(t) = a_2(t)z_2(t) + b_2(t),\ z_2(t_2^*) = 1.$$

As shown in [7] , the solution x_* of (1) is asymptotically stable with respect to admissible perturbations of initial data if the number $\gamma = u(t_1^*)z_1(t_2^*)z_2(t_0 + T) + v(t_1^*)z_1(t_2^*)$ satisfies $|\gamma| < 1$.

5 Example: fold bifurcation of periodic solution

In construction of the example we follow [8] where also the physics of different terms and the role of positive and negative feedbacks were discussed.

Consider the example of equation (1) of the form

$$(x + Px)' = (1 - \sin t)e^{-(x+1)^2} + \lambda - e^{-0.0625}(0.5x + 1.625) \tag{5}$$

with the scalar parameter λ. Assume that the Preisach measure density is $\mu(\alpha,\beta) \equiv 0.125$ in the triangle $-4 < \beta < \alpha < 0$ and is zero outside this triangle. Figure 1(a) shows periodic solutions of equation (5) for $\lambda = 0.2$: all of them have one maximum and one minimum on the period $[0, 2\pi]$.

Numerical simulation shows that for any λ from the interval $0.155 < \lambda < 0.255$ equation (5) has three periodic solutions: two asymptotically stable and one unstable. The S-shaped curves on Figure 1(c) show the maximum and the minimum value of the three periodic solutions as functions of λ. Figure 1(d) presents the value of γ evaluated for these solutions: $0 < \gamma < 1$ for both the stable solutions and $\gamma > 1$ for the unstable solution. At the fold bifurcation points γ approaches the value 1 from below and above for the colliding pair.

6 Periodic solutions with multiple local extrema

Now we adapt the above algorithm of linear stability analysis to solutions with multiple minima on a period. To illustrate the required modification it suffices to consider a T-periodic solution $x_*(t)$ of (1) which has four local extrema: a local minimum x_1^* at a point t_1^*, a local maximum x_2^* at a point t_2^*, a global minimum x_4^* at a point t_4^* and a global maximum x_5^* at a point t_5^* with $t_0 < t_1^* < t_2^* < t_4^* < t_5^* < t_0 + T$. Assume that the right and left second derivatives of $x_*(t)$ are positive at the points t_1^*, t_4^* and negative at $t = t_2^*, t_5^*$. Define the point $t_3^* \in (t_2^*, t_4^*)$ where $x_*(t_3^*) = x_1^*$ and denote

$$\bar{f}_{n1}(t) = \widetilde{f}_x(t, x_*(t), x_5^*), \quad \bar{f}_{n2}(t) = \widetilde{f}_{\beta_0}(t, x_*(t), x_5^*), \quad n = 1, 4, 6$$
$$\bar{f}_{n1}(t) = \hat{f}_x(t, x_*(t), x_{n-1}^*), \quad \bar{f}_{n2}(t) = \hat{f}_{\alpha_0}(t, x_*(t), x_{n-1}^*), \quad n = 2, 5$$
$$\bar{f}_{31}(t) = \widetilde{f}_x(t, x_*(t), x_2^*), \quad \bar{f}_{32}(t) = \widetilde{f}_{\beta_0}(t, x_*(t), x_2^*);$$

the point t_3^* is important because the right hand side of equation (3) is $\widetilde{f}(t, x_*(t), x_2^*)$ for $t_2^* < t < t_3^*$ and $\widetilde{f}(t, x_*(t), x_5^*)$ for $t_3^* < t < t_4^*$. Now consider the auxiliary linear problems

$$\begin{aligned}
u_n'(t) &= \bar{f}_{n1}(t) u_n(t), && u_n(t_{n-1}^*) = 1, && n = 1, 4,\\
v_n'(t) &= \bar{f}_{n1}(t) v_n(t) + \bar{f}_{n2}(t), && v_n(t_{n-1}^*) = 0, && n = 1, 4,\\
z_n'(t) &= \bar{f}_{n1}(t) z_n(t) + \bar{f}_{n2}(t), && z_n(t_{n-1}^*) = 1, && n = 2, 3, 5, 6,
\end{aligned}$$

where $t_0^* = t_0$. Using their solutions of this problems and the quantity $J_3 = J_\beta(x_2^*, x_5^*)/(a + J_\beta(x_1^*, x_5^*))$, we define the number

$$\tilde{\gamma} = z_5(t_5^*)\big(v_4(t_4^*) + (z_3(t_3^*) z_2(t_2^*)(1 - J_3) + J_3) u_4(t_4^*)(u_1(t_1^*) z_6(t_0 + T) + v(t_1^*))\big).$$

Theorem 6.1. *If* $|\tilde{\gamma}| < 1$*, then the periodic solution* x_* *of* (1) *is asymptotically stable with respect to admissible perturbations of initial data.*

The effect of the local extrema is that for a short period of time near the moment t_3^* the periodic solution x_* and any its small perturbation satisfy to different equations (3): the difference is close to a non-zero constant. This generates the term J_3 in the expression for $\tilde{\gamma}$, which is not present in the expression for the quantity γ of Section **4** for the simpler solutions.

For example, consider the equation

$$(x + Px)' = (1 - 0.5 \sin t - \cos 2t) e^{-(x+1)^2} + \lambda - e^{-0.0625}(0.5x + 1.625),$$

where the density of the Preisach measure is $\mu = 0.5$ in the triangle $-2 < \beta < \alpha < 0$ and $\mu = 0$ outside this triangle. The graphs of the three periodic solutions for $\lambda = 0.071$ are shown in Figure 1(b). Here $\tilde{\gamma} = 0.986$ for the topmost stable periodic solution and $\tilde{\gamma} = 1.095$ for the unstable solution, $J_3 = 0.166$. We note that the effect of local extrema is important in this example, since we obtain $\tilde{\gamma} = 1.009$ for the stable solution by setting $J_3 = 0$.

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Rate independent processes in viscous solids

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Abstract

Coupling of rate-independent processes with rate-dependent ones (and thus so-called energetic solutions with weak solutions) is illustrated on so-called generalized standard solids (of Halphen-Nguen type) at small strains. The former processes may involve plasticity, damage, or phase transformations, while the considered rate dependent phenomena are viscosity and possibly also inertia.

Keywords: Energetic solution, weak solution, generalized standard solids.

1 Activated processes in generalized standard materials

Theory of rate-independent processes based on so-called energetic solution by Mielke at al. [16, 17, 18] has been extensively developed and widely applied in [1, 2, 4, 5, 7, 8, 9, 11, 12, 13, 14]. Coupling rate-independent processes with some others that are rate dependent bring, in general, serious difficulties, cf. e.g. [3, 6]. In some cases when such processes are coupled rather indirectly such combination is, however, well possible, cf. [15] for a special model of damage.

The goal of this contribution is to illustrate it briefly on a so-called *generalized standard solids* combined with a *viscous-like response* in a Kelvin-Voigt-type rheology and inertia at *small strains*. We allow for a so-called *gradient theory* as far as internal parameters concern. Altogether, we thus have in mind the following system

$$\varrho\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\big(\sigma_{\rm vi}+\sigma_{\rm el}\big) = f, \quad \sigma_{\rm el} = \varphi_e'(e(u),z,\nabla z), \quad \sigma_{\rm vi} = \zeta_2'\big(e(\frac{\partial u}{\partial t})\big), \tag{1a}$$

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \sigma_{\rm in} \ni 0, \qquad \sigma_{\rm in} = \varphi_z'(e(u),z,\nabla z) - \mathrm{div}\,\varphi_{\nabla z}'(e(u),z,\nabla z), \tag{1b}$$

where $u:\Omega\to\mathbb{R}^n$ is a displacement, $e(u):=\frac{\nabla u^\top+\nabla u}{2}$ the small-strain tensor, $z:\Omega\to\mathbb{R}^m$ a vector of specific internal parameters, $\varrho\geq 0$ mass density,

$\varphi : \mathbb{R}^{n\times n} \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to [0, +\infty]$ is a stored energy being a function of the small-strain tensor e and the vector z and its spatial gradient, and $\zeta_2 : \mathbb{R}^{n\times n} \to [0, +\infty)$ and $\zeta_1 : \mathbb{R}^m \to [0, +\infty]$ are (pseudo)potentials of dissipative forces. From φ, one derives the "elastic" stress $\sigma_{\rm el}$ and an "inelastic" driving force $\sigma_{\rm in}$ as said in (1). Such z may involve plastic strain, hardening, damage, or volume fractions in various phase transformations, etc. We will assume ζ_1 *homogeneous of degree 1* and ζ_2 homogeneous of degree 2 and even just quadratic which is just responsible for the linear viscous-like response. Thus we consider

$$\zeta_1(\dot z) := \delta^*_S(\dot z) \quad \text{with } S \subset \mathbb{R}^m \text{ convex closed,} \qquad \zeta_2(\dot e) := \frac{1}{2}\mathbb{D}\dot e{:}\dot e, \tag{2}$$

where δ^*_S is the Legendre-Fenchel conjugate function to the indicator function δ_S of S and $\mathbb{D} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is a 4th-order tensor (assumed symmetric positive definite). Assuming S bounded (resp. containing 0 in its interior) makes ζ_1 coercive (resp. bounded). Also, $S = \partial\zeta_1(0)$. Nonsmoothness of ζ_1 at 0, which follows from its homogeneity of degree 1 (except the trivial case where ζ_1 is linear), may describe various *activated processes*, i.e. to trigger z evolving, the driving force $\varphi'_z(e(u), z, \nabla z)$ must exceed a certain activation threshold, namely the boundary of S. We further assume φ of 2-growth and coercive on $\mathrm{dom}(\zeta_1)$, beside (8) below.

We assume the body to occupy a domain $\Omega \subset \mathbb{R}^n$ where the system (1) is to hold, and consider it completed by the boundary conditions, say of the Dirichlet/Neumann-type $u = 0$ and $\varphi'_{\nabla z}\nu = 0$ with ν a normal to $\partial\Omega$.

2 Concept of combination of weak and energetic solutions

The inclusion (1b) can advantageously treated when exploiting the degree-1 homogeneity of ζ_1. By this homogeneity and by (2), we have

$$\mathrm{div}\,\varphi'_{\nabla z}(e(u), z, \nabla z) - \varphi'_z(e(u), z, \nabla z) = -\sigma_{\rm in} \in \partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) \subset \partial\zeta_1(0) = S. \tag{3}$$

Due to the gradient term, we must understand it functionally. Introducing $R_1(z) := \int_\Omega \zeta_1(z(x))\mathrm{d}x$ and $V(e, z) := \int_\Omega \varphi(e(u(x)), z(x), \nabla z(x))\mathrm{d}x$ and counting the Neumann boundary conditions and the definition of the subdifferential $\partial R_1(0)$ and properties of ζ_1, (3) means $0 = R_1(0) \le R_1(v) + \langle V'_z(e(u), z), v\rangle$ for any $v \in W^{1,2}(\Omega;\mathbb{R}^m)$. Written it for $v - z$ instead of v, we have $0 \le R_1(v-z) + \langle V'_z(e(u), z), v-z\rangle$. If assuming $\varphi(e, \cdot, \cdot)$ convex, we have $V(e, z) \le V(e, v) - \langle V'_z(e, z), v-z\rangle$. Altogether, at a current time level t (with x-dependence omitted for brevity) for $e = e(u)$, we have

$$\int_\Omega \varphi\big(e(u(t)), z(t), \nabla z(t)\big)\,\mathrm{d}x \le \int_\Omega \varphi\big(e(u(t)), v, \nabla v\big) + \zeta_1\big(v-z(t)\big)\,\mathrm{d}x \tag{4}$$

for all $v \in W^{1,2}(\Omega;\mathbb{R}^m)$. If $z(t)$ satisfies (4), we say that z is *semi-stable* at t; the adjective "semi" distinguishes (4) from (12) below.

We will consider an initial-value problem for (1), namely the initial conditions

$$u(0,\cdot) = u_0, \quad \frac{\partial u}{\partial t}(0,\cdot) = \dot u_0, \quad z(0,\cdot) = z_0. \tag{5}$$

For a fixed time horizon $T > 0$, we denote $I = (0,T)$, $\bar I = [0,T]$, $Q = I \times \Omega$. Notation of function spaces L^p and $W^{1,p}$ is standard. Further, we denote by $\mathrm{B}(\bar I; X)$ the Banach space of bounded measurable functions $\bar I \to X$; note that these functions are defined everywhere on $\bar I$. Likewise, $\mathrm{BV}(\bar I; X)$ will stand for functions with bounded variations. The variation $\mathrm{Var}_S(z;0,T)$ of the process $z : I \to L^1(\Omega;\mathbb{R}^m)$ over $\bar I$ with respect to the norm $\int_\Omega \delta_S^*(\cdot)\,\mathrm{d}x$ is defined as $\mathrm{Var}_S(z;0,T) := \sup \sum_{i=1}^k \int_\Omega \delta_S^*\big(z(t_i,x){-}z(t_{i-1},x)\big)\,\mathrm{d}x$ where the supremum is taken over all partitions of the type $0 = t_0 < t_1 < ... < t_k = T$, $k \in \mathbb{N}$. Note that, if $\frac{\partial z}{\partial t} \in L^1(Q;\mathbb{R}^m)$, then $\mathrm{Var}_S(z;0,T) = \int_Q \delta_S^*(\frac{\partial z}{\partial t})\,\mathrm{d}x\mathrm{d}t$. In general, we do not expect $\frac{\partial z}{\partial t} \in L^1(Q;\mathbb{R}^m)$, however.

Definition 2.1. *We call (u,z) with $u \in W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n))$ and $z \in \mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m)) \cap \mathrm{B}(\bar I;W^{1,2}(\Omega;\mathbb{R}^m))$ an energetic solution to the problem (1) with the above mentioned initial/boundary conditions*

(i) *if (1a) holds in the weak sense, i.e. for $v \in C^1(Q;\mathbb{R}^n)$ with $v(T,\cdot) = 0$:*

$$\int_Q \Big(\sigma_{\rm vi}{+}\sigma_{\rm el}\Big){:}\nabla v - \varrho\frac{\partial u}{\partial t}{\cdot}\frac{\partial v}{\partial t} - f{\cdot}v\,\mathrm{d}x\mathrm{d}t = \int_\Omega \varrho \dot u_0{\cdot}v(0,\cdot)\,\mathrm{d}x \tag{6}$$

where $\sigma_{\rm el} = \varphi_e'(e(u),z,\nabla z)$ and $\sigma_{\rm vi} = \zeta_2'(e(\frac{\partial u}{\partial t}))$ as in (1a), and

(ii) *if the energy inequality holds, i.e.*

$$\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 \mathrm{d}x + V\big(e(u(T)),z(T)\big) + \mathrm{Var}_S(z;0,T) + 2\int_Q \zeta_2\big(e(\frac{\partial u}{\partial t})\big)\mathrm{d}x\mathrm{d}t$$
$$\le \int_\Omega \frac{\varrho}{2}|\dot u_0|^2\,\mathrm{d}x + V\big(e(u_0),z_0\big) + \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t \tag{7}$$

with $V(e,z) := \int_\Omega \varphi(e(u(x)),z(x),\nabla z(x))\,\mathrm{d}x$ as above, and

(iii) *if the semi-stability (4) holds for all $v \in W^{1,2}(\Omega;\mathbb{R}^m)$ and $t \in I$,*

(iv) *the initial conditions $u(0,\cdot) = u_0$ and $z(0,\cdot) = z_0$ are satisfied.*

Proposition 2.2. *Let $\zeta_1 : \mathbb{R}^m \to [0,+\infty]$ be convex coercive homogeneous degree-1, $u_0 \in W^{1,2}(\Omega;\mathbb{R}^n)$, $\dot u_0 \in L^2(\Omega;\mathbb{R}^n)$, $z_0 \in W^{1,2}(\Omega;\mathbb{R}^m)$, $f \in L^2(Q;\mathbb{R}^n)$, $\varphi(e,z,\nabla z) \ge \varepsilon(|e|^2 + |\nabla z|^2)$, $\varphi(\cdot,z,\cdot)$ convex and, if $\varrho > 0$, then even φ convex, and let one of the two cases hold:*

$$\varphi(e,z,\nabla z) = \varphi_1(e,z) + \varphi_2(e,z,\nabla z), \qquad \varphi_1 \text{ continuous}, \tag{8a}$$
$$\varphi(e,z,\nabla z) = \varphi_3(e,z) + \varphi_4(z,\nabla z), \qquad \varphi(e,\cdot,\cdot) \text{ quadratic}, \tag{8b}$$
$$\varphi(e,z,\nabla z) = \varphi_3(e,z) + \varphi_4(z,\nabla z), \qquad \zeta_1 \text{ continuous}, \tag{8c}$$

with $\varphi_1(\cdot,z)$ affine, φ_2 quadratic, and $\varphi_3(\cdot,z)$ uniformly convex. Then the energetic solution to (1) with (5) exist.

Sketch of the proof. The implicit time discretization (called also a backward Euler formula) of (1) gives an approximate solution; it is important that on each level, the incremental problem has a variational structure so that its solution exists by a direct method, cf. [19]. A-priori estimates for thus approximated u in $L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^n))\cap W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n))$ and $\varrho\frac{\partial u}{\partial t}$ in $W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n)^*)$ and z in $L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^m))\cap\mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))$ can be derived. Also, a discrete variant of (7) can be obtained if φ is convex or $\varrho=0$. Convergent subsequences are then chosen by Banach's and Helly's selection principles. Strong convergence of $e(u)$ in $L^2(Q;\mathbb{R}^n)$ in case (8b,c) allows for the limit passage in (6), while in case (8a) it is simply due to weak continuity. The limit passage in (4) is by a so-called binomial trick in case (8a,b) or by lower semicontinuity and (generalized) Aubin-Lions' compact embedding $L^\infty(I;W^{1,2}(\Omega))\cap\mathrm{BV}(\bar I;L^1(\Omega))\subset L^1(Q)$ in case (8c). Eventually, the limit passage in (7) is by weak lower-semicontinuity. □

It is important that Definition **2**.1 contains indeed all information that allows to pass back to the original problem if formulated weakly, cf. [19]. Moreover, though it was derived under convexity assumption for $\varphi(e,\cdot,\cdot)$, Definition **2**.1 itself works without this assumption, too. Note also that the limit passage in (4) is, in general, more difficult than for the usual "full" stability, see (12). In addition, one can derive equality in (7) by using (4).

Applications compatible with (8a) includes linearized plasticity with hardening (even without ∇z) or, with (8a,b), even some nonlinear version of it. Applications of (8b,c) are some special models for shape-memory alloys and magneto- or electro-strictive materials.

3 Infinitesimally slow loadings and rate-independent limits

An interesting question is how solutions to (1) behave for very slow loading regimes f. By proper scaling of time like εt on the fixed time interval $[0,T]$, we can replace ϱ by $\varepsilon^2\varrho$ and ζ_2 by $\varepsilon\zeta_2$ while ζ_1, φ, and f remain unchanged. Let $(u_\varepsilon,z_\varepsilon)$ denote the corresponding energetic solution. By the method used before, one can easily get a-priori estimates of $(u_\varepsilon,z_\varepsilon)$:

$$\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^n))}\le\frac{C}{\varepsilon},\tag{9a}$$

$$\Big\|e\Big(\frac{\partial u_\varepsilon}{\partial t}\Big)\Big\|_{L^2(Q;\mathbb{R}^{n\times n})}\le\frac{C}{\sqrt{\varepsilon}},\tag{9b}$$

$$\big\|e(u_\varepsilon)\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{n\times n}))}\le C,\tag{9c}$$

$$\big\|z_\varepsilon\big\|_{L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^m))\cap\mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))}\le C.\tag{9d}$$

Let us investigate convergence for $\varepsilon\to0$ (in terms of subsequences) to some (u,z). Using $|\int_Q\varepsilon\zeta_2'(e(\frac{\partial u_\varepsilon}{\partial t})):e(v)\mathrm{d}x\mathrm{d}t|=\mathcal{O}(\sqrt{\varepsilon})$ and $|\int_Q\varepsilon^2\varrho\frac{\partial u_\varepsilon}{\partial t}\cdot\frac{\partial v}{\partial t}\mathrm{d}x\mathrm{d}t|=\mathcal{O}(\varepsilon)$ we can pass to the limit in the (weakly formulated) force equilibrium: $\int_Q\varepsilon\zeta_2'(e(\frac{\partial u_\varepsilon}{\partial t})):e(v)-\varepsilon^2\varrho\frac{\partial u_\varepsilon}{\partial t}\cdot\frac{\partial v}{\partial t}+\varphi_e'(e(u_\varepsilon),z_\varepsilon):e(v)-f\cdot v\,\mathrm{d}x\mathrm{d}t=0$ to obtain

$\int_Q \varphi_e'(e(u),z):e(v) - f\cdot v\,\mathrm{d}x\mathrm{d}t = 0$. Convergence for $\varepsilon \to 0$ in the semi-stability $\int_\Omega \varphi(e(u_\varepsilon(t)), z_\varepsilon(t), \nabla z_\varepsilon(t))\,\mathrm{d}x \le \int_\Omega \varphi(e(u_\varepsilon(t)), v, \nabla v) + \zeta_1(v - z_\varepsilon(t))\,\mathrm{d}x$ is as before (namely by "binomial trick" relying on φ quadratic). As we now loose any estimate on $\frac{\partial u}{\partial t}$, we must qualify f better, namely $f \in W^{1,1}(I; L^2(\Omega;\mathbb{R}^n))$, and use the by-part integration in time $\int_Q f\cdot\frac{\partial u}{\partial t}\mathrm{d}x\mathrm{d}t = \int_\Omega f(T)\cdot u(T) - f(0)\cdot u_0\,\mathrm{d}x - \int_Q \frac{\partial f}{\partial t}\cdot u\,\mathrm{d}x\mathrm{d}t$ so that the Helmholtz-type energy inequality (7) written for u_ε turns into the Gibbs-type energy inequality

$$\int_\Omega \frac{\varepsilon^2\varrho}{2}\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2\mathrm{d}x + G\big(T, u_\varepsilon(T), z_\varepsilon(T)\big) + 2\int_Q \varepsilon\zeta_2\Big(e\big(\frac{\partial u_\varepsilon}{\partial t}\big)\Big)\mathrm{d}x\mathrm{d}t$$
$$+ \mathrm{Var}_S(z_\varepsilon; 0, T) \le \int_\Omega \frac{\varepsilon^2\varrho}{2}|\dot u_0|^2\,\mathrm{d}x + G\big(0, u_0, z_0\big) - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,\mathrm{d}x\mathrm{d}t \qquad (10)$$

where now $G(t,u,z) := \int_\Omega \varphi(e(u), z, \nabla z) - f(t)\cdot u\,\mathrm{d}x$. The limit passage in (10) is simple just by omitting the terms $\frac{\varepsilon^2\varrho}{2}|\frac{\partial u_\varepsilon}{\partial t}(T)|^2$ and $\varepsilon\zeta_2(e(\frac{\partial u_\varepsilon}{\partial t}))$, and using weak lower-semicontinuity, so that we obtain

$$G\big(T, u(T), z(T)\big) + \mathrm{Var}_S(z; 0, T) \le G\big(0, u_0, z_0\big) - \int_Q \frac{\partial f}{\partial t}\cdot u\,\mathrm{d}x\mathrm{d}t. \qquad (11)$$

Thus we can see that, in the "slow-loading" limit, the inertial as well as viscous effects indeed disappear, i.e. we put $\varrho = 0$ and $\zeta_2 = 0$, and the whole system becomes fully *rate independent*. Assuming strict convexity of $\varphi(\cdot, z, \nabla z)$, one can derive so-called full *stability* in the sense

$$G\big(t, u(t), z(t)\big) \le G\big(t, w, v\big) + \int_\Omega \zeta_1\big(v - z(t)\big)\,\mathrm{d}x \qquad (12)$$

to hold for any $(w,v) \in W^{1,2}(\Omega;\mathbb{R}^n\times\mathbb{R}^m)$. The relations (11)–(12) define a (now standard) *energetic solution* invented by Mielke at al. [16, 17, 18].

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