Optimal transport and Fokker-Planck equations with monotone drifts

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Hammamet, *May 4, 2010*

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Measure-valued solutions to Fokker-Planck equations

We consider nonnegative measure-valued solutions to the Fokker-Planck equation

$$\boxed{\partial_t \rho - \Delta \rho - \nabla \cdot (\rho B) = 0, \quad \rho\big|_{t=0} = \rho_0,} \tag{FP}$$

where $B : \mathbb{R}^d \to \mathbb{R}^d$ is a Borel $\boldsymbol{\lambda}$-monotone operator, $\boldsymbol{\lambda} \in \mathbb{R}$, i.e.

$$\langle B(x) - B(y), x - y \rangle \geq \boldsymbol{\lambda} \left| x - y \right|^2 \quad \text{for every } x, y \in \mathbb{R}^d.$$

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- No growth restrictions are imposed on B.

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In particular, it has been shown that ρ_t **is absolutely continuous** with respect to the Lebesgue measure for $\mathscr{L}^1$-a.e. t.

Main problem

Here we want to study the stability properties of the distributional solutions of

$$\partial_t \rho - \Delta\rho - \nabla \cdot (\rho B) = 0 \qquad \text{(FP)}$$

in terms of the *transportation costs*

$$\mathcal{C}_h(\boldsymbol{\rho}^1, \boldsymbol{\rho}^2) := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} h(|\mathbf{x}_1 - \mathbf{x}_2|) \, \mathrm{d}\boldsymbol{\rho}(\mathbf{x}_1, \mathbf{x}_2) : \boldsymbol{\rho} \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d), \right.$$
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If $\boldsymbol{\rho}^1, \boldsymbol{\rho}^2$ are two distributional solutions to (FP) find suitable estimates of

$$\mathcal{C}_h(\boldsymbol{\rho}^1_{\mathbf{t}}, \boldsymbol{\rho}^2_{\mathbf{t}}) \textit{ in terms of } \mathcal{C}_h(\boldsymbol{\rho}^1_{\mathbf{0}}, \boldsymbol{\rho}^2_{\mathbf{0}}).$$

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The Wasserstein approach to Fokker-Planck in the gradient case

When

$B = \nabla V$ is the **gradient** of a **λ**-convex function $V : \mathbb{R}^d \to \mathbb{R}$

then (FP) can be considered as the **gradient flow** of the relative entropy functional

$$\begin{aligned} \mathrm{Ent}_\gamma(\rho) &:= \int_{\mathbb{R}^d} u(x) \log u(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} V(x)\,\mathrm{d}\rho(x) && \rho = u\mathscr{L}^d \\ &= \int_{\mathbb{R}^d} \frac{\mathrm{d}\rho}{\mathrm{d}\gamma} \log\Big(\frac{\mathrm{d}\rho}{\mathrm{d}\gamma}\Big)\,\mathrm{d}\gamma && \gamma := \mathrm{e}^{-V}\mathscr{L}^d \end{aligned}$$

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in the space $\mathscr{P}_2(\mathbb{R}^d)$ of probability measures with finite quadratic moments endowed with the so called L^2-Wasserstein distance $\mathrm{W}_2(\cdot,\cdot)$

$$\mathrm{W}_2^2(\rho^1,\rho^2) := \mathcal{C}_h(\rho^1,\rho^2), \quad h(r) := r^2.$$

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This remarkable interpretation found by Jordan-Kinderlehrer-Otto (1998) gave rise to a series of studies on the relationships between certain classes of diffusion equations and distances between probability measures induced by optimal transport problems.
General overviews can be found in the books Villani 2003-2009, Ambrosio-Gigli-S. 2005.

Wasserstein distance (I): the point of view of optimal transport

Wasserstein space:

$$\mathscr{P}_2(\mathbb{R}^d) := \Big\{\rho \text{ Borel probability measures on } \mathbb{R}^d \text{ s.t. } \int_{\mathbb{R}^d} |x|^2 \,\mathrm{d}\rho(x) < +\infty\Big\}.$$

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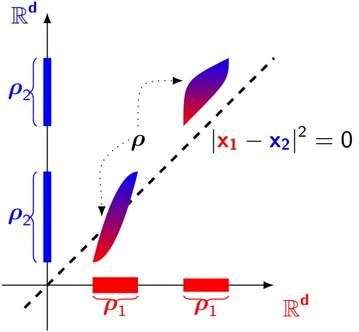
Couplings between $\rho_1, \rho_2 \in \mathscr{P}_2(\mathbb{R}^d)$: measures $\rho \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ whose marginals are ρ_1, ρ_2, i.e.

$$\rho(A \times \mathbb{R}^d) = \rho_1(A), \quad \rho(\mathbb{R}^d \times B) = \rho_2(B)$$

for every Borel sets $A, B \subset \mathbb{R}^d$.
$\Gamma(\rho_1, \rho_2)$ is the collection of all couplings of ρ_1 and ρ_2.



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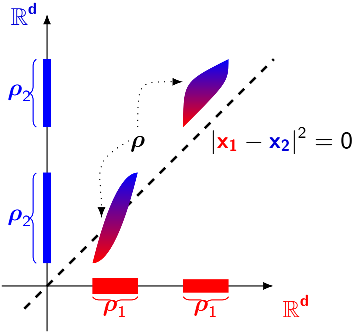

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“Soft” properties

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- **Convexity** (but linear segments are not geodesics!)
- Existence of (constant speed, minimizing) **geodesics** connecting arbitrary measures $\rho_0, \rho_1 \in \mathscr{P}_2(\mathbb{R}^d)$: they are curves $\rho : s \in [0,1] \mapsto \rho_s$ s.t.

$$\mathrm{W}_2(\rho_0, \rho_1) = \mathcal{L}_0^1[\rho], \quad \mathrm{W}_2(\rho_{s_0}, \rho_{s_1}) = |s_1 - s_0|\, \mathrm{W}_2(\rho_0, \rho_1), \quad 0 \le s_0 \le s_1 \le 1.$$

Wasserstein distance (II): the Brenier dynamical approach

In $\mathbb{R}^d$ consider a moving family $\rho_s \in \mathscr{P}_2(\mathbb{R}^d)$, $s \in [0,1]$, satisfying the **continuity equation**

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Wasserstein distance W_2 between ρ_0 and ρ_1:

$$\mathrm{W}_2(\rho_0,\rho_1) := \min\Big\{\mathcal{L}_0^1[\rho] : \rho\big|_{s=0} = \rho_0,\ \rho\big|_{s=1} = \rho_1\Big\}.$$

Otto's approach to Gradient flows through Wasserstein distance

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho\, \mathbf{v}\right) = 0 & \textbf{(Continuity equation)} \\ \mathbf{v} = -\mathrm{D}\left(\dfrac{\boldsymbol{\delta\Phi}}{\boldsymbol{\delta\rho}}\right) & \textbf{(Nonlinear condition)} \\ \rho\big|_{t=0} = \rho_0 & \rho_0 \in \mathscr{P}_2(\mathbb{R}^d). \end{cases}$$

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[Otto, with Jordan and Kinderlehrer '97-'00] showed in many interesting cases that such kind of equations can be interpreted as

> the **"gradient flow"** of Φ with respect to the **Wasserstein distance** W_2 in $\mathscr{P}_2(\mathbb{R}^d)$.

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$$\boxed{\mathrm{W}_2(\rho^1_t, \rho^2_t) \le \mathrm{e}^{-\lambda t}\, \mathrm{W}_2(\rho^1_0, \rho^2_0) \qquad \text{for every } t \ge 0,}$$

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1. Derivative of the **squared distance function** along locally Lipschitz curves.
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3. **Subgradient inequality** for the logarithmic gradient $\nabla\rho/\rho + \nabla V$

Outline

Derivative of the squared distance function

The first tool is the formula which evaluates the derivative of the squared Wasserstein distance from a fixed measure σ along the locally Lipschitz curve ρ in $\mathscr{P}_2(\mathbb{R}^d)$.

Moreover

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\rho_t,\sigma) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \mathbf{v}_t(x), x-y\rangle \,\mathrm{d}\boldsymbol{\rho}_t(x,y) \quad \textit{for } \mathscr{L}^1\textit{-a.e. } t>0$$

where $\boldsymbol{\rho}_t$ is an optimal coupling between ρ_t and σ.

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Theorem

Let $\rho : (0, +\infty) \to \mathscr{P}_2(\mathbb{R}^d)$ be a locally Lipschitz curve. Then for $\mathscr{L}^1$-a.e. $t > 0$ there exists a (ρ_t-essentially) unique Borel map $\mathbf{v}_t \in L^2(\rho_t; \mathbb{R}^d)$ such that

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and

$$\lim_{h \to 0} \frac{\mathrm{W}_2(\rho_{t+h}, \rho_t)}{|h|} = \Big(\int_{\mathbb{R}^d} |\mathbf{v}_t|^2 \, \mathrm{d}\rho_t \Big)^{1/2} \in L^\infty_{\mathrm{loc}}(0, +\infty).$$

$\mathbf{v}_t$ *belongs to* $\mathrm{Tan}_{\rho_t} \mathscr{P}_2(\mathbb{R}^d)$, *the closure in* $L^2(\rho_t; \mathbb{R}^d)$ *of the* **space generated by the gradients of $C^\infty_c(\mathbb{R}^d)$ functions.**

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Displacement convexity

Definition (Displacement λ-convex, McCann '97)

A functional $\Phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ is **displacement convex** if each couple of measures $\rho_0, \rho_1 \in D(\Phi)$ can be connected by a **geodesic** (i.e. displacement interpolation) $\rho_s,\ s \in [0,1]$ such that

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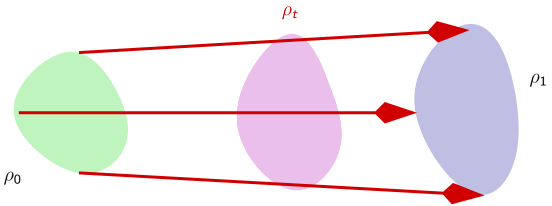

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Definition (Displacement λ-convex, McCann '97)

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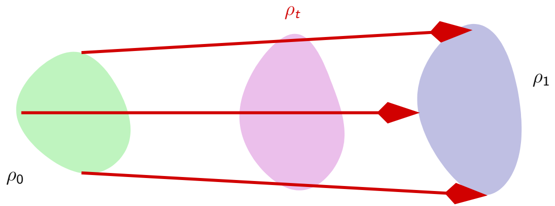

Displacement convexity of the relative entropy functional

Consider

$$\begin{aligned}\Phi(\rho) = \mathrm{Ent}_\gamma(\rho) &:= \int_{\mathbb{R}^d} u(x)\log u(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} V(x)\,\mathrm{d}\rho(x) && \rho = u\mathscr{L}^d\\ &= \int_{\mathbb{R}^d} \frac{\mathrm{d}\rho}{\mathrm{d}\gamma}\log\left(\frac{\mathrm{d}\rho}{\mathrm{d}\gamma}\right)\mathrm{d}\gamma && \gamma := \mathrm{e}^{-V}\mathscr{L}^d\end{aligned}$$

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Theorem (McCann '97)

The relative entropy functional $\mathrm{Ent}_\gamma(\cdot)$ *is* $\boldsymbol{\lambda}$*-displacement convex iff*

the potential V **is** $\boldsymbol{\lambda}$**- convex**, *i.e.* $D^2 V(x)\, \xi \cdot \xi \geq \boldsymbol{\lambda} |\xi|^2 \qquad \forall x, \xi \in \mathbb{R}^d$.

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When $\boldsymbol{\lambda} = 0$ one can give an equivalent characterization in terms of $\gamma = \mathrm{e}^{-V}\mathscr{L}^d$ which holds even in infinite dimensional Hilbert spaces H:

Theorem (Ambrosio-Gigli-S. '05)

The relative entropy functional Ent_γ *is displacement convex iff* γ **is log-concave**

$$\gamma((1-s)A + sB) \geq \gamma(A)^{1-s}\gamma(B)^s \quad \forall\, A, B \text{ open subset of } H, \quad s \in [0,1].$$

Subgradient characterization of the (FP)-velocity field

The last ingredient is the "subgradient" property of the vector field $\mathbf{v}$ given by the "logarithmic gradient" of ρ

$$\mathbf{v} = \frac{\nabla\rho}{\rho} + \nabla V$$

if and only if there exists a (unique) vector field $\mathbf{v} \in \mathrm{Tan}_\rho \mathscr{P}_2(\mathbb{R}^d) \subset L^2(\rho; \mathbb{R}^d)$ *such that*

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The functional I_γ is called **relative Fisher information**.

Wasserstein solutions to (FP)

If $\rho : (0, +\infty) \to \mathscr{P}_2(\mathbb{R}^d)$ is a locally Lipschitz curve solving the Fokker-Planck equation

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and $\boldsymbol{\rho}_t$ is an optimal coupling between ρ_t and $\sigma \in \mathscr{P}_2(\mathbb{R}^d)$ we have

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Wasserstein solutions to (FP)

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$$\partial_t \rho_t + \mathrm{div}(\rho_t \mathbf{v}_t) = 0, \quad -\mathbf{v}_t = \Big(\frac{\nabla \rho_t}{\rho_t} + \nabla V\Big), \tag{FP}$$

and $\boldsymbol{\rho}_t$ is an optimal coupling between ρ_t and $\sigma \in \mathscr{P}_2(\mathbb{R}^d)$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\rho_t,\sigma) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle -\mathbf{v}_t(x), y-x\rangle \,\mathrm{d}\boldsymbol{\rho}_t(x,y) \\ &\le \mathrm{Ent}_\gamma(\sigma) - \mathrm{Ent}_\gamma(\rho) - \frac{1}{2}\boldsymbol{\lambda}\,\mathrm{W}_2^2(\rho,\sigma)\end{aligned}$$

Definition

A *locally Lipschitz* curve $\rho : (0,+\infty) \to \mathscr{P}_2(\mathbb{R}^d)$ is a "Wasserstein" solution to (FP) if it satisfies the Evolution Variational Inequality

$$\boxed{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{W}_2^2(\rho_t,\sigma) \le \mathrm{Ent}_\gamma(\sigma) - \mathrm{Ent}_\gamma(\rho_t) - \frac{1}{2}\boldsymbol{\lambda}\,\mathrm{W}_2^2(\rho_t,\sigma)} \tag{EVI}$$

for every $\sigma \in \mathscr{P}_2(\mathbb{R}^d)$.

Proof of contraction

Let ρ^1, ρ^2 be locally Lipschitz curves solving (EVI) ($\lambda = 0$)

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$$\frac{1}{2}\frac{\partial}{\partial r} \mathrm{W}_2^2(\boldsymbol{\rho}^1_{\mathsf{r}}, \boldsymbol{\rho}^2_{\mathsf{s}}) \leq \mathrm{Ent}_\gamma(\boldsymbol{\rho}^2_{\mathsf{s}}) - \mathrm{Ent}_\gamma(\boldsymbol{\rho}^1_{\mathsf{r}})$$

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Summing up

$$\frac{1}{2}\frac{\partial}{\partial r}\mathrm{W}_2^2(\rho^1_r, \rho^2_s) + \frac{1}{2}\frac{\partial}{\partial s}\mathrm{W}_2^2(\rho^1_r, \rho^2_s) \le 0$$

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and evaluating the inequality at $t = r = s$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{W}_2^2(\rho^1_t, \rho^2_t) = \frac{1}{2}\Big(\frac{\partial}{\partial r}\mathrm{W}_2^2(\rho^1_r, \rho^2_s) + \frac{\partial}{\partial s}\mathrm{W}_2^2(\rho^1_r, \rho^2_s)\Big)_{t=r=s} \leq 0$$

A general existence result

Here H is a separable Hilbert space.

Theorem (Ambrosio-Gigli-S. '05, S. '07)

If $\Phi : \mathscr{P}_2(H) \to (-\infty, +\infty]$ *is a proper, lower semicontinuous, and displacement* $\boldsymbol{\lambda}$*-convex functional then for every initial measure* $\rho_0 \in \overline{D(\Phi)} \subset \mathscr{P}_2(\mathbb{R}^d)$ *there exists a unique locally Lipschitz solution to*

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In the case $H = \mathbb{R}^d$ when $\Phi = \mathrm{Ent}_\gamma$, ρ is a distributional solution to

$$\partial_t\rho - \Delta\rho - \mathrm{div}(\rho\nabla V) = 0 \quad \text{(FP)}$$

and satisfies the regularization property

$$\mathrm{I}_\gamma(\rho_t) = \int_{\mathbb{R}^d} \Big|\frac{\nabla\rho_t}{\rho_t} + \nabla V\Big|^2 \mathrm{d}\rho_t \le C\,t^{-2}\,\mathrm{e}^{2\boldsymbol{\lambda} t}$$

Stability: an example in infinite dimension

Let H be an infinite dimensional Hilbert space and $\gamma \in \mathscr{P}(H)$ be a **log-concave** non degenerate measure.

Take a family of finite n-dimensional spaces H^n with a log-concave measure $\gamma_n = e^{-V^n} \cdot \mathscr{L}^n|_{H^n} \in \mathscr{P}(H^n)$ and suppose that

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On each finite dimensional space H^n we are solving

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Theorem [Ambrosio-S.-Zambotti]

If $\rho_0^n \to \rho_0$ in $\mathscr{P}_2(H)$ then $\rho_t^n \to \rho_t$ in $\mathscr{P}_2(H)$, and ρ_t is the unique solution of the gradient flow generated by Ent_γ in $\mathscr{P}_2(X)$ starting from ρ_0.

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Application:

The Dirichlet form $\int_H \|\nabla u\|_H^2 \, \mathrm{d}\gamma$ is **closable.**

Outline

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- We cannot directly differentiate the Wasserstein distance.

General transportation costs

We consider a function

$$h : [0, +\infty) \to [0, +\infty) \textit{ continuous and } \textbf{non-decreasing},\ h(0) = 0,$$

and an associated cost function $c(\mathbf{x_1}, \mathbf{x_2}) := h\big(|\mathbf{x_1} - \mathbf{x_2}|\big)$.

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$$\mathcal{C}_h(\boldsymbol{\rho}^1,\boldsymbol{\rho}^2) := \inf\Big\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} h\big(|\mathbf{x_1}-\mathbf{x_2}|\big)\,\mathrm{d}\boldsymbol{\rho}(\mathbf{x_1},\mathbf{x_2}) : \boldsymbol{\rho}\in\mathscr{P}(\mathbb{R}^d\times\mathbb{R}^d),$$

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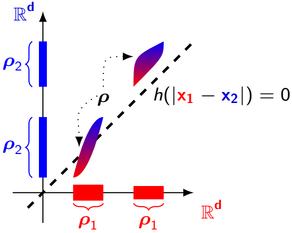

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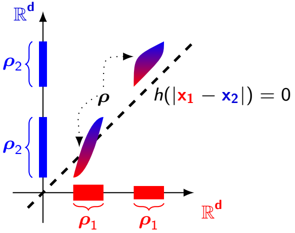

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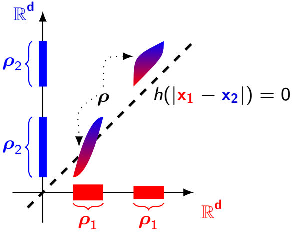

Interesting cases:

- $h(r) = r^p$, $\mathcal{C}_h(\rho^1, \rho^2)$ induces the p-Wasserstein distance W_p^p
- h bounded, concave, strictly increasing: $\mathcal{C}_h(\rho^1, \rho^2)$ is a metric on $\mathscr{P}(\mathbb{R}^d)$ inducing weak convergence.

Main result

Assume that $B : \mathbb{R}^d \to \mathbb{R}^d$ is a Borel monotone drift

then

$$\mathcal{C}_{h_{\lambda_t}}(\rho^1_t, \rho^2_t) \le \mathcal{C}_h(\rho^1_0, \rho^2_0) \quad \textit{for every } t \ge 0.$$

In the p-homogeneous case when $h(r) := r^p$ *we have*

$$\mathrm{W}_p(\rho^1_t, \rho^2_t) \le \mathrm{e}^{-\lambda t}\mathrm{W}_p(\rho^1_0, \rho^2_0).$$

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Theorem (Natile-Peletier-S. '10)

If ρ^1, ρ^2 are two distributional solutions to (FP)

$$\partial_t \rho - \Delta\rho - \nabla \cdot (\rho B) = 0 \tag{FP}$$

satisfying the summability condition

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^d} |B(x) \qquad | \, \mathrm{d}\rho_t(x) \, \mathrm{d}t < +\infty \quad \textit{for every } 0 < t_0 < t_1 < +\infty,$$

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In the p-homogeneous case when $h(r) := r^p$ we have

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Assume that $B : \mathbb{R}^d \to \mathbb{R}^d$ is a Borel $\boldsymbol{\lambda}$-monotone drift and denote by $h_{\boldsymbol{\lambda} t}$ the rescaled cost

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$$\mathcal{C}_{h_{\boldsymbol{\lambda} t}}(\boldsymbol{\rho}^1_{\mathbf{t}}, \boldsymbol{\rho}^2_{\mathbf{t}}) \le \mathcal{C}_h(\boldsymbol{\rho}^1_{\mathbf{0}}, \boldsymbol{\rho}^2_{\mathbf{0}}) \quad \textit{for every } t \ge 0.$$

In the p-homogeneous case when $h(r) := r^p$ we have

$$\mathrm{W}_p(\boldsymbol{\rho}^1_{\mathbf{t}}, \boldsymbol{\rho}^2_{\mathbf{t}}) \le \mathrm{e}^{-\boldsymbol{\lambda} t}\mathrm{W}_p(\boldsymbol{\rho}^1_{\mathbf{0}}, \boldsymbol{\rho}^2_{\mathbf{0}}).$$

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- **Variable-doubling** technique and **maximum principle** to get refined comparison results for smooth solutions of the backward equation.

Kantorovich duality

Our argument strongly relies on the dual Kantorovich representation of $\mathcal{C}_h(\cdot,\cdot)$

$$\mathcal{C}_h(\rho^1,\rho^2) = \sup\Big\{ \int_{\mathbb{R}^d} \psi^1(\mathsf{x}_1)\,\mathrm{d}\rho^1(\mathsf{x}_1) + \int_{\mathbb{R}^d} \psi^2(\mathsf{x}_2)\,\mathrm{d}\rho^2(\mathsf{x}_2) : \psi^1,\psi^2 \in C_\mathrm{b}(\mathbb{R}^d),\quad \psi^1(\mathsf{x}_1)+\psi^2(\mathsf{x}_2) \le h(|\mathsf{x}_1-\mathsf{x}_2|)\Big\}$$

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Kantorovich duality reduces the estimate of the cost $\mathcal{C}_h(\rho^1_\mathsf{T},\rho^2_\mathsf{T})$ of two solutions of (FP) at a certain final time T to the estimate of

$$\Sigma(\psi^1,\psi^2;T) := \int_{\mathbb{R}^d} \psi^1\,\mathrm{d}\rho^1_\mathsf{T} + \int_{\mathbb{R}^d} \psi^2\,\mathrm{d}\rho^2_\mathsf{T}$$

for an arbitrary pair of functions ψ^1,ψ^2 satisfying the constraint

$$\psi^1(\mathsf{x}_1)+\psi^2(\mathsf{x}_2) \le h(|\mathsf{x}_1-\mathsf{x}_2|) \quad \text{for every } \mathsf{x}_1,\mathsf{x}_2 \in \mathbb{R}^d.$$

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Assume for a moment that B is **bounded and smooth**. In order to estimate

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we can **solve the final-value problem for the adjoint backward Kolomogorov equation**

$$\partial_t \psi^{1,2} + \Delta \psi^{1,2} - B \cdot \nabla \psi^{1,2} = 0 \quad \text{in } \mathbb{R}^d \times (0, T), \quad \psi^{1,2}(\cdot, T) := \psi^{1,2} \quad \text{(BKE)}$$

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then we would get the *uniform bound in term of the initial cost*

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4 regularization steps

- **Regularization of h**: we can assume that

$$h \text{ is } C^1, \text{ Lipschitz}, \lim_{r\uparrow+\infty} h(r) = +\infty, \text{ and } \mathcal{C}_h(\rho_0^1, \rho_0^2) < +\infty$$

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Combine the Fitzpatrick-Phelps approximation of B by bounded montone maps with convolution.

A comparison result for the backward equation

Theorem

Let

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- $\psi^1, \psi^2 \in C^{2,1}_{\mathrm{b}}(\mathbb{R}^d \times [0, T])$ *be solutions of the "backward" inequality*

$$\partial_t \psi + \Delta \psi - B_t \cdot \nabla \psi \geq 0 \quad \text{in } \mathbb{R}^d \times [0, T]$$

such that

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Then

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The maximum-principle argument

If ψ^1, ψ^2 satisfy the *strict* inequality $\boxed{\partial_t\psi + L\psi > 0}$, $L\psi := \Delta\psi - B\cdot\nabla\psi$,
then the function

$$f(\mathbf{x}_1, \mathbf{x}_2, t) := \psi^1(\mathbf{x}_1, s) + \psi^2(\mathbf{x}_2, t) - h(|\mathbf{x}_1 - \mathbf{x}_2|)$$

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$$\boxed{B(\bar{\mathbf{x}}_1)\cdot\nabla_{\mathbf{x}_1}\psi^1(\bar{\mathbf{x}}_1, \bar{t}) + B(\bar{\mathbf{x}}_2)\cdot\nabla_{\mathbf{x}_2}\psi^2(\bar{\mathbf{x}}_2, \bar{t}) = \bar{h}\left(B(\bar{\mathbf{x}}_1) - B(\bar{\mathbf{x}}_2)\right)\cdot(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \ge 0}$$

On the other hand, the function

$$\mathbb{R}^d \ni z \mapsto \psi^1(\bar{\mathbf{x}}_1 + z, \bar{t}) + \psi^2(\bar{\mathbf{x}}_2 + z, \bar{t}) - H(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2) = f(\bar{\mathbf{x}}_1 + z, \bar{\mathbf{x}}_2 + z, \bar{t})$$

has a local maximum at $z = 0$ so that

The maximum-principle argument

If ψ^1, ψ^2 satisfy the *strict* inequality $\boxed{\partial_t\psi + L\psi > 0}$, $L\psi := \Delta\psi - B\cdot\nabla\psi$, then the function

$$f(\mathsf{x}_1, \mathsf{x}_2, t) := \psi^1(\mathsf{x}_1, s) + \psi^2(\mathsf{x}_2, t) - h(|\mathsf{x}_1 - \mathsf{x}_2|)$$

cannot attain a (local) maximum in a point $(\bar{\mathsf{x}}_1, \bar{\mathsf{x}}_2, \bar t)$ with $\bar t < T$.

By contradiction, if $(\bar{\mathsf{x}}_1, \bar{\mathsf{x}}_2, \bar t)$ is a **local maximizer of f** with $\bar t < T$ then

$$\partial_t f(\bar{\mathsf{x}}_1, \bar{\mathsf{x}}_2, \bar t) \le 0, \quad \nabla_{\mathsf{x}_1} f(\bar{\mathsf{x}}_1, \bar{\mathsf{x}}_2, \bar t) = 0, \quad \nabla_{\mathsf{x}_2} f(\bar{\mathsf{x}}_1, \bar{\mathsf{x}}_2, \bar t) = 0;$$

$$\boxed{\partial_t\psi^1(\bar{\mathsf{x}}_1, \bar t) + \partial_t\psi^2(\bar{\mathsf{x}}_2, \bar t) \le 0}$$

$$\nabla_{\mathsf{x}_1}\psi^1(\bar{\mathsf{x}}_1, \bar t) = \bar h\,(\bar{\mathsf{x}}_1 - \bar{\mathsf{x}}_2) = -\nabla_{\mathsf{x}_2}\psi^2(\bar{\mathsf{x}}_2, \bar t), \quad \bar h := \frac{h'(|\bar{\mathsf{x}}_1 - \bar{\mathsf{x}}_2|)}{|\bar{\mathsf{x}}_1 - \bar{\mathsf{x}}_2|} \ge 0$$

It follows that

$$\boxed{B(\bar{\mathsf{x}}_1)\cdot\nabla_{\mathsf{x}_1}\psi^1(\bar{\mathsf{x}}_1, \bar t) + B(\bar{\mathsf{x}}_2)\cdot\nabla_{\mathsf{x}_2}\psi^2(\bar{\mathsf{x}}_2, \bar t) = \bar h\,\big(B(\bar{\mathsf{x}}_1) - B(\bar{\mathsf{x}}_2)\big)\cdot(\bar{\mathsf{x}}_1 - \bar{\mathsf{x}}_2) \ge 0}$$

On the other hand, the function

$$\mathbb{R}^d \ni z \mapsto \psi^1(\bar{\mathsf{x}}_1 + z, \bar t) + \psi^2(\bar{\mathsf{x}}_2 + z, \bar t) - H(\bar{\mathsf{x}}_1, \bar{\mathsf{x}}_2) = f(\bar{\mathsf{x}}_1 + z, \bar{\mathsf{x}}_2 + z, \bar t)$$

has a local maximum at $z = 0$ so that $\boxed{\Delta_{\mathsf{x}_1}\psi^1(\bar{\mathsf{x}}_1, \bar t) + \Delta_{\mathsf{x}_2}\psi^2(\bar{\mathsf{x}}_2, \bar t) \le 0.}$

The maximum-principle argument

If ψ^1, ψ^2 satisfy the *strict* inequality $\boxed{\partial_t \psi + L\psi > 0}$, $L\psi := \Delta\psi - B\cdot\nabla\psi$, then the function

$$f(\mathbf{x}_1, \mathbf{x}_2, t) := \psi^1(\mathbf{x}_1, s) + \psi^2(\mathbf{x}_2, t) - h(|\mathbf{x}_1 - \mathbf{x}_2|)$$

cannot attain a (local) maximum in a point $(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{t})$ with $\bar{t} < T$.

By contradiction, if $(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{t})$ is a **local maximizer of f** with $\bar{t} < T$ then

$$\partial_t f(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{t}) \le 0, \quad \nabla_{\mathbf{x}_1} f(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{t}) = 0, \quad \nabla_{\mathbf{x}_2} f(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{t}) = 0;$$

$$\boxed{\partial_t \psi^1(\bar{\mathbf{x}}_1, \bar{t}) + \partial_t \psi^2(\bar{\mathbf{x}}_2, \bar{t}) \le 0}$$

$$\nabla_{\mathbf{x}_1} \psi^1(\bar{\mathbf{x}}_1, \bar{t}) = \bar{h}\,(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = -\nabla_{\mathbf{x}_2} \psi^2(\bar{\mathbf{x}}_2, \bar{t}), \quad \bar{h} := \frac{h'(|\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2|)}{|\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2|} \ge 0$$

It follows that

$$\boxed{B(\bar{\mathbf{x}}_1)\cdot\nabla_{\mathbf{x}_1}\psi^1(\bar{\mathbf{x}}_1, \bar{t}) + B(\bar{\mathbf{x}}_2)\cdot\nabla_{\mathbf{x}_2}\psi^2(\bar{\mathbf{x}}_2, \bar{t}) = \bar{h}\,\big(B(\bar{\mathbf{x}}_1) - B(\bar{\mathbf{x}}_2)\big)\cdot(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \ge 0}$$

On the other hand, the function

$$\mathbb{R}^d \ni z \mapsto \psi^1(\bar{\mathbf{x}}_1 + z, \bar{t}) + \psi^2(\bar{\mathbf{x}}_2 + z, \bar{t}) - H(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2) = f(\bar{\mathbf{x}}_1 + z, \bar{\mathbf{x}}_2 + z, \bar{t})$$

has a local maximum at $z = 0$ so that $\boxed{\Delta_{\mathbf{x}_1}\psi^1(\bar{\mathbf{x}}_1, \bar{t}) + \Delta_{\mathbf{x}_2}\psi^2(\bar{\mathbf{x}}_2, \bar{t}) \le 0.}$

Combining the previous inequality we obtain the contradiction

$$(\partial_t \psi^1 + L\psi^1)(\bar{\mathbf{x}}_1, \bar{t}) + (\partial_t \psi^2 + L\psi^2)(\bar{\mathbf{x}}_2, \bar{t}) \le 0.$$

Open problems

- Find a “natural” Wasserstein-metric notion of solution.
- Prove suitable regularization estimates
- Prove general Trotter-Kato approximation formula for the solution
- Extend the theory to operators defined in a bounded domain
- Extend the theory to the infinite-dimensional case.
- ....